Geometry of the quantum space

J. Alvarez, A. Dector and J. David Vergara
Instituto de Ciencias Nucleares, UNAM

JHEP03(2017)044, JMP7(2016)1627.

Mexicuerdas.07.08.2017

Contents

Introduction

- Geometry in the parameter space

- Classical Hamiltonian systems

- Classical Hamiltonian systems are defined by specifying the Hamiltonian $H(p,q)$ satisfying the canonical relations

$$\{q^i, p_j\} = \delta^i_j,$$

- We will mostly be interested in families of canonical transformations that depend continuously on some parameters α_a.

- These continuous canonical transformation are generated by functions A_a

$$q^i(\alpha+\delta\alpha) = q^i(\alpha) + \frac{\partial q^i}{\partial \alpha^a}\delta\alpha^a = q^i(\alpha) - \frac{\partial A_a}{\partial p_i}\delta\alpha^a = q^i(\alpha) - \{q^i, A_a\}\delta\alpha^a,$$

$$p_i(\alpha+\delta\alpha) = p_i(\alpha) + \frac{\partial p_i}{\partial \alpha^a}\delta\alpha^a = p_i(\alpha) + \frac{\partial A_a}{\partial q^i}\delta\alpha^a = p_i(\alpha) - \{p_i, A_a\}\delta\alpha^a.$$

Then we have the relations

$$\frac{\partial q^i}{\partial \alpha^a} = -\frac{\partial A_a}{\partial p_i}, \qquad \frac{\partial p_i}{\partial \alpha^a} = \frac{\partial A_a}{\partial q^i}.$$

In the case of time translations

$$q^i(t+dt) = q^i(t) + \frac{dq^i}{dt}dt = q^i(t) - \frac{\partial A_t}{\partial p_i}dt = q^i(\alpha) + \left\{q^i, H\right\}dt,$$

$$p_i(t+dt) = p_i(\alpha) + \frac{dp_i}{dt}dt = p_i(t) + \frac{\partial A_t}{\partial q^i}dt = p_i(\alpha) + \left\{p_i, H\right\}dt.$$

then in this case the generator is

$$A_t = -H$$

Canonical transformations are conventionally expressed through generating functions $F_1(q_0, q, \alpha)$ such that

$$p^i{}_0 = \frac{\partial F_1}{\partial q^i{}_0}, \quad p_i(\alpha) = -\frac{\partial F_1}{\partial q^i(\alpha)}.$$

Differentiating the second equation with respect to α at constant $q(\alpha)$ we find

$$\frac{\partial p_i(\alpha)}{\partial \alpha^a} = \frac{\partial A_a}{\partial q^i(\alpha)} = -\frac{\partial^2 F}{\partial q^i(\alpha)\partial \alpha^a}.$$

Clearly this equation can be satisfied if we choose

$$A_a = -\frac{\partial F}{\partial \alpha^a}$$

Notice that in this case our potentials are pure gauge, since they have vanishing curvature,

$$F_{ab} = \partial_a A_b - \partial_b A_a = 0$$

to get a non vanishing curvature we must consider a nonholonomic mapping.

Quantum Information Metric and Berry Curvature

Let us consider a quantum system in a state $|\psi\rangle$

Furthermore, let us assume that the system's Hamiltonian depends explicitly on a number of real parameters λ^a $a = 1,\ldots N$

Let us now consider that the system undergoes a small change in some or all of the parameters in the form $\lambda^a + \delta\lambda^a$

So that the original state changes $|\psi(\lambda)\rangle \rightarrow |\psi(\lambda+\delta\lambda)\rangle$

We will now focus on computing the fidelity $\mathcal{F}$, that is defined as

$$\mathcal{F}(\lambda,\lambda+\delta\lambda) = |\langle\psi(\lambda+\delta\lambda)|\psi(\lambda)\rangle|,$$

physically it represents a measure of the change of the state under changes in its parameter space. This quantity gets divergent at quantum critical points and thus can be used as an order parameter of quantum phase transitions.

If we now make a Taylor expansion of $|\psi(\lambda+\delta\lambda)\rangle$ we obtain

$$|\psi(\lambda+\delta\lambda)\rangle = |\psi(\lambda)\rangle + |\partial_a\psi(\lambda)\rangle\delta\lambda^a + \frac{1}{2}|\partial_a\partial_b\psi(\lambda)\rangle\delta\lambda^a\delta\lambda^b + \cdots,$$

so that

$$\langle\psi(\lambda)|\psi(\lambda+\delta\lambda)\rangle = 1 + \langle\psi(\lambda)|\partial_a\psi(\lambda)\rangle\delta\lambda^a + \frac{1}{2}\langle\psi(\lambda)|\partial_a\partial_b\psi(\lambda)\rangle\delta\lambda^a\delta\lambda^b + \cdots.$$

then for the fidelity we obtain

$$\mathcal{F}(\lambda,\lambda+\delta\lambda) = 1 + \frac{1}{2}G_{ab}\,\delta\lambda^a\delta\lambda^b + \cdots,$$

where G_{ab} is the complex Quantum Geometric Tensor, given by

$$G_{ab} = \langle\partial_a\psi|\partial_b\psi\rangle - \langle\partial_a\psi|\psi\rangle\langle\psi|\partial_b\psi\rangle.$$

We note that

$$\mathbb{R}\langle\psi|\partial_a\psi\rangle=\frac{1}{2}\left(\langle\psi|\partial_a\psi\rangle+\langle\psi|\partial_a\psi\rangle^*\right)=\frac{1}{2}\left(\langle\psi|\partial_a\psi\rangle+\langle\partial_a\psi|\psi\rangle\right)=\frac{1}{2}\partial_a\langle\psi|\psi\rangle=0.$$

Therefore we can write

$$\langle\psi|\partial_a\psi\rangle\equiv -iA_a$$

with $A_a\in\mathbb{R}.$ This is now our gauge potential.

The quantum geometric tensor can be divided in its real an imaginary part.

Taking its real part we obtain the Quantum information metric

$$g_{ab}\equiv\mathbf{Re}\,G_{ab}=\frac{1}{2}\left(\langle\partial_a\psi|\partial_b\psi\rangle+\langle\partial_b\psi|\partial_a\psi\rangle\right)-\langle\partial_a\psi|\psi\rangle\langle\psi|\partial_b\psi\rangle=$$

$$=\frac{1}{2}\left(\langle\partial_a\psi|\partial_b\psi\rangle+\langle\partial_b\psi|\partial_a\psi\rangle\right)-A_aA_b.$$

This metric is gauge invariant, under gauge transformations in the parameter space

$$\psi' = \exp\left(i\alpha(\lambda)\right)\psi,$$

However, could be not gauge invariant if the gauge transformation is valued in the physical space

$$\psi' = \exp\left(i\alpha(\lambda, x)\right)\psi,$$

To see this we rewrite the Quantum Information metric in the form

$$g_{ab} = \mathbb{R}e\left[\left\langle\left(\partial_a - iA_a\right)\psi \mid \left(\partial_a - iA_a\right)\psi\right\rangle\right]$$

where we notice that

$$\left(\partial_a - iA_a\right)\psi = D_a\psi.$$

Then to define a gauge invariant Quantum Information metric, we extend the definition of the gauge potential in such way, always transform as

$$\Gamma'_a = \Gamma_a + \partial_a \alpha,$$

in this way the covariant derivative transforms like

$$\left(D_a \psi\right)' = \exp\left(i\alpha\right)\left(D_a \psi\right),$$

under any change of gauge. Using the covariant derivative the QIM takes the form

$$g_{ab} = \mathbb{R}e\left\langle D_a \psi \mid D_b \psi \right\rangle = \mathbb{R}e\int d^3x \left(D_a \psi\right)^* \left(D_b \psi\right).$$

This expression will be gauge invariant under any kind of gauge transformation.

On the other hand, the imaginary part of G_{ab} will yield the Berry curvature F_{ab} given by

$$\frac{1}{2}F_{ab} \equiv \mathbf{Im}\, G_{ab} = \frac{1}{2i}\left(\langle \partial_a \psi \mid \partial_b \psi \rangle - \langle \partial_b \psi \mid \partial_a \psi \rangle\right) = \frac{1}{2}\left\{\partial_a A_b - \partial_b A_a\right\}$$

The Berry connection is directly related to the Berry phase of the system, given as

$$\gamma(\mathcal{C}) = \oint_{\mathcal{C}} A,$$

The Berry phase describes a cyclic adiabatic evolution along $\mathcal{C}$. By making use of Stokes theorem, we can rewrite the Berry phase as

$$\gamma(\mathcal{C}) = \int_{\Sigma} F,$$

where $\partial\Sigma = \mathcal{C}$.

- A Lagrangian approach to quantum information metric and the Berry curvature

We start by considering a quantum field theory defined by a Lagrangian

$$\mathcal{L}_0(\lambda^a)$$

during the Euclidean time interval $(-\infty, 0)$. Then consider the situation when a deformation of the original theory is turned on at time $\tau = 0$.

In this manner, we obtain a perturbed Lagrangian $\mathcal{L}_1$ for the remaining Euclidean time interval $(0, \infty)$, given explicitly by

$$\mathcal{L}_1 = \mathcal{L}_0 + \delta\lambda^a \mathcal{O}_a \qquad (2)$$

where the $\mathcal{O}_a$ are deformation operators which are functions of the theory's degrees of freedom.

Both the original and deformed theories have corresponding ground states

$$|\Omega_0\rangle \quad \text{and} \quad |\Omega_1\rangle$$

We will now focus on computing the fidelity of this system, defined here as the absolute square of the overlap between both states at temporal infinity

$$\mathcal{F}(\lambda, \lambda + \delta\lambda) \equiv | \langle \Omega_1, \tau \to \infty | \Omega_0, \tau \to -\infty \rangle$$

Physically speaking, the fidelity thus defined will give us a measure of the change effectuated on the system by turning on the deformation terms.

By considering any generic state $\left|\varphi\right\rangle$ and its overlap with the original state $\left|\Omega_0\right\rangle$

we built the path integral

$$\langle \varphi | \Omega_0 \rangle = \frac{1}{\sqrt{Z_0}} \int_{\varphi(\tau=0)=\tilde{\varphi}} \mathcal{D}\varphi \exp\left(-\int_{-\infty}^{0} d\tau \int d^d x \, \mathcal{L}_0 \right),$$

where

$$Z_0 = \int \mathcal{D}\varphi \exp\left(-\int_{-\infty}^{\infty} d\tau \int d^d x \, \mathcal{L}_0 \right).$$

Similarly, one can consider the evolution from $\tau = 0$ where the state $|\varphi\rangle$ is inserted, to $\tau = \infty$ where we are placing the perturbed state $|\Omega_1\rangle$
In the path integral-formalism this is given by

$$\langle \Omega_1 | \varphi \rangle = \frac{1}{\sqrt{Z_1}} \int_{\varphi(\tau=0)=\tilde{\varphi}} \mathcal{D}\varphi \exp\left(-\int_0^\infty d\tau \int d^d x \, \mathcal{L}_1 \right) =$$

$$= \frac{1}{\sqrt{Z_1}} \int_{\varphi(\tau=0)=\tilde{\varphi}} \mathcal{D}\varphi \exp\left(-\int_0^\infty d\tau \int d^d x \left(\mathcal{L}_0 + \delta\lambda^a \mathcal{O}_a \right) \right),$$

where

$$Z_1 = \int \mathcal{D}\varphi \exp\left(-\int_{-\infty}^\infty d\tau \int d^d x \left(\mathcal{L}_0 + \delta\lambda^a \mathcal{O}_a \right) \right).$$

Thus, the overlap between both states is given by

$$\langle \Omega_1 | \Omega_0 \rangle = \int \mathcal{D}\varphi \, \langle \Omega_1 | \varphi \rangle \langle \varphi | \Omega_0 \rangle =$$

$$= \frac{1}{\sqrt{Z_0 Z_1}} \int \mathcal{D}\varphi \, \exp\left(-\int_{-\infty}^{0} d\tau \int d^d x \, \mathcal{L}_0 - \int_0^{\infty} d\tau \int d^d x \left(\mathcal{L}_0 + \delta\lambda^a \mathcal{O}_a \right) \right) =$$

$$= \frac{1}{\sqrt{Z_0 Z_1}} \int \mathcal{D}\varphi \, \exp\left(-\int_{-\infty}^{\infty} d\tau \int d^d x \, \mathcal{L}_0 \right) \exp\left(-\int_0^{\infty} d\tau \int d^d x \, \delta\lambda^a \mathcal{O}_a \right) = \frac{Z_2}{\sqrt{Z_0 Z_1}} \, .$$

Where Z_2 is the partition function of a field theory that is deformed only for $\tau > 0,$ which can be written more conveniently as

$$\langle \Omega_1 | \Omega_0 \rangle = \frac{\left\langle \exp\left(-\int_0^{\infty} d\tau \int d^d x \, \delta\lambda^a \mathcal{O}_a \right) \right\rangle}{\left\langle \exp\left(-\int_{-\infty}^{\infty} d\tau \int d^d x \, \delta\lambda^a \mathcal{O}_a \right) \right\rangle^{1/2}} \, ,$$

where the expectation value is taken with respect to the unperturbed state.

We can now take the above expression to expand $|\langle\Omega_1|\Omega_0\rangle|$ in a series of $\delta\lambda^a$.

The final result of this expansion of the fidelity can be written as

$$|\langle\Omega_1|\Omega_0\rangle| = 1 - \frac{1}{2} G_{ab}\,\delta\lambda^a\delta\lambda^b + \cdots.$$

where the quantity G_{ab} is the complex quantum geometric tensor and is given here by

$$G_{ab} = \int_{-\infty}^{0} d\tau_1 \int_0^{\infty} d\tau_2 \int d^d x_1 \int d^d x_2 \Big(\langle\mathcal{O}_a(\tau_1,x_1)\mathcal{O}_b(\tau_2,x_2)\rangle - \langle\mathcal{O}_a(\tau,x_1)\rangle\langle\mathcal{O}_b(\tau,x_2)\rangle\Big).$$

where there is an implicit time ordering in the two-point function, given that this expression is obtained from a path-integral approach.

Quantum Mechanical Systems: The Quantum Information Metric and the Berry Curvature

We now consider the particular case of a D=0+1 quantum field theory, i.e. a quantum mechanical system.

Therefore, in the case of quantum mechanical systems the quantum geometric tensor reduces to the form

$$G_{ab} = \int_{-\infty}^{0} d\tau_1 \int_{0}^{\infty} d\tau_2 \Big(\langle \mathcal{O}_a(\tau_1) \mathcal{O}_b(\tau_2) \rangle - \langle \mathcal{O}_a(\tau) \rangle \langle \mathcal{O}_b(\tau) \rangle \Big).$$

In analogy with the standard formulation of the quantum geometric tensor, we intend to divide the above expression in its real and imaginary parts.

The real part can be written as

$$\mathbf{Re}\,\mathrm{G}_{\mathrm{ab}}=\int_{-\infty}^{0}\mathrm{d}\tau_1\int_0^{\infty}\mathrm{d}\tau_2\left(\frac{1}{2}\Big(\langle\mathcal{O}_{\mathrm{a}}(\tau_1)\mathcal{O}_{\mathrm{b}}(\tau_2)\rangle+\langle\mathcal{O}_{\mathrm{b}}(\tau_1)\mathcal{O}_{\mathrm{a}}(\tau_2)\rangle\Big)-\langle\mathcal{O}_{\mathrm{a}}(\tau_1)\rangle\langle\mathcal{O}_{\mathrm{b}}(\tau_2)\rangle\right)=$$

$$=\int_{-\infty}^{0}d\tau_1\int_0^{\infty}d\tau_2\left(\frac{1}{2}\langle\{\mathcal{O}_a(\tau_1),\mathcal{O}_b(\tau_2)\}\rangle-\langle\mathcal{O}_a(\tau_1)\rangle\langle\mathcal{O}_b(\tau_2)\rangle\right),$$

whereas the imaginary part takes the form

$$\mathbf{Im}\,\mathrm{G}_{\mathrm{ab}}=\frac{1}{2\mathrm{i}}\int_{-\infty}^{0}\mathrm{d}\tau_1\int_0^{\infty}\mathrm{d}\tau_2\Big(\langle\mathcal{O}_{\mathrm{a}}(\tau_1)\mathcal{O}_{\mathrm{b}}(\tau_2)\rangle-\langle\mathcal{O}_{\mathrm{b}}(\tau_1)\mathcal{O}_{\mathrm{a}}(\tau_2)\rangle\Big)=\frac{1}{2\mathrm{i}}\int_{-\infty}^{0}\mathrm{d}\tau_1\int_0^{\infty}\mathrm{d}\tau_2\,\langle[\mathcal{O}_{\mathrm{a}}(\tau_1),\mathcal{O}_{\mathrm{b}}(\tau_2)]\rangle,$$

We then make the claim that the real part corresponds to the Quantum Information Metric

$$g_{ab}=\int_{-\infty}^{0}d\tau_1\int_0^{\infty}d\tau_2\left(\frac{1}{2}\langle\{\mathcal{O}_a(\tau_1),\mathcal{O}_b(\tau_2)\}\rangle-\langle\mathcal{O}_a(\tau_1)\rangle\langle\mathcal{O}_b(\tau_2)\rangle\right),\quad (3)$$

while the imaginary part corresponds to the Berry Curvature

$$\mathbf{Im}\, G_{ab} = \frac{1}{2} F_{ab}$$

$$F_{ab} = \frac{1}{i}\int_{-\infty}^{0} d\tau_1 \int_{0}^{\infty} d\tau_2 \, \langle [\mathcal{O}_a(\tau_1), \mathcal{O}_b(\tau_2)] \rangle . \qquad (4)$$

Conformal Field Theory (Miyaji, et al Phys.Rev.Lett.115(2015)261602)

Using the two point function for a primary operator

$$\left\langle\left(\mathcal{O}(\tau_1,x_1)\mathcal{O}(\tau_2,x_2)\right)\right\rangle = \frac{C}{\left(\left(\tau_1-\tau_2\right)^2+\left(x_1-x_2\right)^2\right)^{\Delta}}$$

if $d+1-2\Delta<0$ we get:

$$G_{\lambda\lambda} = CN_d V_{R^d}\varepsilon^{d+2-2\Delta}$$

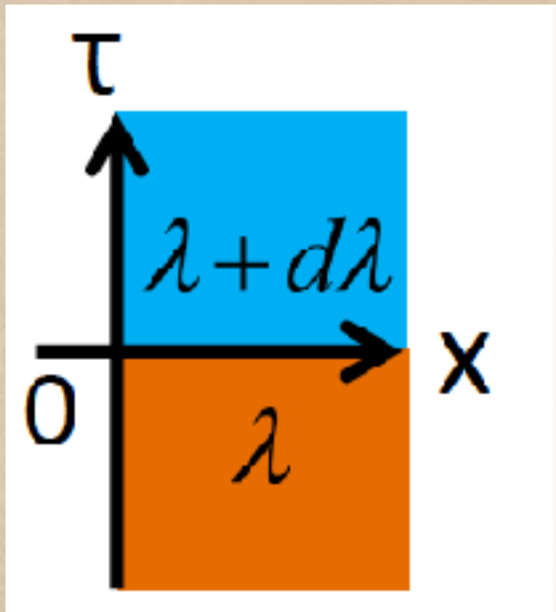
with

$$N_d = \frac{2^{d-2\Delta}\pi^{(d)/2}\Gamma\left(\Delta-d/2-1\right)}{\left(2\Delta-d-1\right)\Gamma\left(\Delta\right)}$$

A Gravity Dual Proposal of Information Metric

We focus on an exactly marginal perturbation i.e. Δ=d+1.

(2+1) Exact Gravity Dual via Janus Solutions
A gravity dual of the CFT with the interface is known as a **Janus solution.**



AdS3 Janus model

$$S_{Janus} = \frac{-1}{16\pi G_N}\int dx^3 \sqrt{g}\left(R - g^{ab}\partial_a\lambda\partial_b\lambda + 2R_{AdS}{}^{-2}\right)$$

$$ds^2 = R_{AdS}{}^2\left(dy^2 + f(y)ds_{AdS2}{}^2\right) \quad \lambda(y) = \gamma\int_{-\infty}^{y}\frac{dy}{f(y)} + \lambda_{-\infty},$$

$$f(y) = \frac{1}{2}\left(1+\sqrt{1-2\lambda^2}\cosh(2y)\right), \quad \lambda_\infty - \lambda_{-\infty} \approx \gamma + O\left(\gamma^3\right)$$

γ is the parameter of Janus deformation.

- In this model we can evaluate the classical on-shell action

$$S_{Janus}(\gamma) - S_{Janus}(0) = \frac{R_{AdS} V_1}{4\pi G_N \varepsilon} \log\left(\frac{1}{1-2\gamma^2}\right) > 0,$$

- where ε is the UV cut off in the AdS2.

- Thus we can estimate the information metric as

$$\left|\langle \Psi(\gamma) | \Psi(0) \rangle\right| = \exp\left(-S_{Janus}(\gamma) + S_{Janus}(0)\right) \approx 1 - \frac{R_{AdS} V_1}{8\pi G_N \varepsilon}\gamma^2,$$

- Then $G_{\lambda\lambda} = \dfrac{cV_1}{12\pi\varepsilon}$ by nothing normalization

$$\lambda_{CFT} \propto \sqrt{c}\,\lambda_{AdS}$$

The Quantum Harmonic Oscillator: Linear Deformation

In order to set notation, we will write the undeformed Hamiltonian H_0 of the system as

$$H_0 = \frac{1}{2}p^2 + \frac{\alpha}{2}q^2 \, ,$$

The simplest deformation we can introduce in Hamiltonian is a term which is linear in the position operator, so that the resulting deformed theory is described by

$$H_1 = \frac{1}{2}p^2 + \frac{\alpha}{2}q^2 + J\,q \, ,$$

Clearly, this is a case where the corresponding deformed Lagrangian can be written in the desired form

$$L_1 = L_0 + J\mathcal{O}_J \text{ ,with } \mathcal{O}_J = -q\,.$$

We should mention that, given that the deformation has only one parameter, we then cannot expect to find a Berry curvature in this example.

The Hamiltonian can be rewritten as

$$H_1 = \frac{1}{2}p^2 + \frac{\alpha}{2}Q^2 - \frac{1}{2}\left(\frac{J}{\sqrt{\alpha}}\right)^2,$$

where

$$Q \equiv q + \frac{J}{\alpha}\,.$$

Let $|\Psi_n^{(0)}\rangle$ and $|\Psi_m^{(1)}\rangle$ be the eigenfunctions of H_0 and H_1 operators, respectively.

with corresponding eigenvalues

$$E_n^{(0)} = \sqrt{\alpha}\left(n + \frac{1}{2}\right), \qquad E_m^{(1)} = \sqrt{\alpha}\left(m + \frac{1}{2}\right) - \frac{1}{2}\left(\frac{J}{\sqrt{\alpha}}\right)^2 .$$

We can now focus on the ground-state functions of each Hamiltonians in the coordinate representation

$$\Psi_0^{(0)}(q) = \left(\frac{\sqrt{\alpha}}{\pi}\right)^{1/4} \mathrm{Exp}\left\{-\frac{\sqrt{\alpha}}{2} q^2\right\},$$

$$\Psi_0^{(1)}(q) = \left(\frac{\sqrt{\alpha}}{\pi}\right)^{1/4} \mathrm{Exp}\left\{-\frac{\sqrt{\alpha}}{2}\left(q + \frac{J}{\alpha}\right)^2\right\}.$$

In this manner, taking the absolute square of the overlap and expanding in series of the parameter J we obtain

$$|\langle \Psi_0^{(1)} | \Psi_0^{(0)} \rangle| = 1 - \frac{1}{4\alpha^{3/2}} J^2 + \cdots,$$

we can now read the quantum information metric

$$g_{JJ} = \frac{1}{2\alpha^{3/2}} . \quad (5)$$

Quantum Information Metric from the Lagrangian Formalism

We can now realize the computation of the quantum information metric using the expression we have obtained from the Lagrangian formalism

$$g_{JJ} = \int_{-\infty}^{0} d\tau_1 \int_{0}^{\infty} d\tau_2 \Big(\langle \Psi_0^{(0)} | \mathcal{O}_J(\tau_1) \mathcal{O}_J(\tau_2) | \Psi_0^{(0)} \rangle - \langle \Psi_0^{(0)} | \mathcal{O}_J(\tau_1) | \Psi_0^{(0)} \rangle \langle \Psi_0^{(0)} | \mathcal{O}_J(\tau_2) | \Psi_0^{(0)} \rangle \Big)$$
$$= \int_{-\infty}^{0} d\tau_1 \int_{0}^{\infty} d\tau_2 \Big(\langle \Psi_0^{(0)} | q(\tau_1) q(\tau_2) | \Psi_0^{(0)} \rangle - \langle \Psi_0^{(0)} | q(\tau_1) | \Psi_0^{(0)} \rangle \langle \Psi_0^{(0)} | q(\tau_2) | \Psi_0^{(0)} \rangle \Big),$$

where the expectation value is taken with respect to the undeformed ground state $|\Psi_0^{(0)}\rangle$ .We first note that the two point function of the quantum harmonic oscillator with respect to its ground state in real time is given by

$$\langle \Psi_0^{(0)} | q(t_1) q(t_2) | \Psi_0^{(0)} \rangle = \frac{1}{i} G_2(t_2 - t_1) = \frac{1}{2\sqrt{\alpha}} \mathrm{Exp}\Big(-i\sqrt{\alpha}\, | t_2 - t_1 | \Big),$$

so that making a Wick rotation we obtain

$$\langle \Psi_0^{(0)} | q(\tau_1) q(\tau_2) | \Psi_0^{(0)} \rangle = \frac{1}{2\sqrt{\alpha}} \mathrm{Exp}\left(-\sqrt{\alpha}(\tau_2 - \tau_1)\right), \tau_2 > \tau_1 ,$$

while one can find $\langle \Psi_0^{(0)} | q | \Psi_0^{(0)} \rangle = 0.$

Therefore, substituting in the expression for the quantum information metric we obtain

$$g_{JJ} = \int_{-\infty}^{0} d\tau_1 \int_{0}^{\infty} d\tau_2 \langle \Psi_0^{(0)} | q(\tau_1) q(\tau_2) | \Psi_0^{(0)} \rangle = \frac{1}{2\sqrt{\alpha}} \int_{-\infty}^{0} d\tau_1 \, e^{\sqrt{\alpha}\tau_1} \int_{0}^{\infty} d\tau_2 \, e^{-\sqrt{\alpha}\tau_2} = \frac{1}{2\alpha^{3/2}},$$

which is exactly the same expression as (5). Therefore, we can see in this example that the application of the Lagrangian formalism can give us the same result for the quantum information metric obtained through direct computation.

The Generalized Harmonic Oscillator

The Hamiltonian of this system is given by

$$H = Zp^2 + Y\{p,q\} + Xq^2 \ ,$$

where X,Y,Z are real numbers that act as the parameters of the system.
To obtain the Lagrangian, we have from Hamilton's equations

$$p = \frac{1}{2Z}(\dot{q} - 2Yq),$$

so that

$$L = \frac{1}{4Z}\dot{q}^2 - \left(X - \frac{Y^2}{Z}\right)q^2 - \frac{Y}{Z}\dot{q}q \, .$$

We can now do small variations of the parameters

$$\{X,Y,Z\} \to \{X+\delta X, Y+\delta Y, Z+\delta Z\}$$

Doing this and writing the deformed Lagrangian in the form (2), we can read the deformation operators as

$$\mathcal{O}_X = -q^2\,, \quad \mathcal{O}_Y = -\frac{q}{2}\left(\dot{q} - 2Yq\right), \quad \mathcal{O}_Z = -\frac{1}{4Z^2}\left(\dot{q} - 2Yq\right)^2 .$$

Note that these operators can be written in term of the momentum as:

$$\mathcal{O}_X = -q^2\,, \quad \mathcal{O}_Y = -2qp = -\left\{q, p\right\}, \quad \mathcal{O}_Z = -p^2\,,$$

as should be expected from inspection of the original Hamiltonian.

The eigenfunctions in coordinate space of the Hamiltonian are given explicitly by

$$\langle q \mid n \rangle = \left(\frac{\Omega}{Z}\right)^{1/4} \chi_n\left(q\sqrt{\frac{\Omega}{Z}}\right) \exp\left(\frac{-iYq^2}{2Z}\right),$$

where we have defined

$$\Omega = \sqrt{XZ - Y^2}\,.$$

where the functions $\chi_n(x)$ are given by

$$\chi_n(x) = \frac{1}{\sqrt{2^n \sqrt{\pi} n!}} e^{-x^2/2} H_n(x),$$

with $\mathrm{H}_n(x)$ the Hermite polynomials. In this manner, the ground state eigenfunction of the system is given by

$$\langle q | 0 \rangle = \left(\frac{\Omega}{\pi Z} \right)^{1/4} \exp\left(-\frac{q^2}{2Z} (\Omega + iY) \right).$$

If we now perform the following canonical transformation

$$q = \sqrt{Z} Q, \qquad p = \frac{1}{Z}(P - YQ),$$

then the Hamiltonian takes the standard form of a harmonic oscillator of frequency Ω

$$\tilde{H} = P^2 + \Omega^2 Q^2,$$

In the case of the generalized harmonic oscillator the quantum information metric with respect to the ground state can be computed exactly and is given by

$$g_{ab} = \frac{1}{32\Omega^4}\begin{pmatrix} Z^2 & -2ZY & 2Y^2 - XZ \\ -2ZY & 4XZ & -2XY \\ 2Y^2 - XZ & -2XY & X^2 \end{pmatrix},$$

Moreover, the generalized harmonic oscillator is a system that exhibits a Berry curvature. Its Berry curvature can be exactly computed from the eigenfunctions of the Hamiltonian. For the case of the ground state it is given explicitly as a differential form as

$$F = \frac{1}{16\Omega^3}\left(XdY \wedge dZ + YdZ \wedge dX + ZdX \wedge dY\right),$$

Quantum Information Metric of the Generalized Harmonic Oscillator from the Lagrangian Formalism.

In the following subsections we will compute the quantum information metric and the Berry curvature from the two-point expressions (3) and (4). We can now express the position and momentum operators Q and P in terms of creation and annihilation operators, we will first obtain the following expectation values

$$\langle 0|\mathcal{O}_X|0\rangle = -\langle 0|q^2|0\rangle = -\frac{Z}{2\Omega}, \quad <0|\mathcal{O}_Y|0> = -\langle 0|\{q,p\}|0\rangle = \frac{Y}{\Omega},$$
$$<0|\mathcal{O}_Z|0> = -\langle 0|p^2|0\rangle = -\frac{X}{2\Omega}.$$

Having calculated these quantities, we can now concentrate on the computation of the two-point functions needed for each metric element.

We will focus on the computation of the g_{XY} element of the quantum information metric,

$$g_{XY} = \int_{-\infty}^{0} d\tau_1 \int_{0}^{\infty} d\tau_2 \left(\frac{1}{2} \langle \{ \mathcal{O}_X(\tau_1), \mathcal{O}_Y(\tau_2) \} \rangle - \langle \mathcal{O}_X(\tau_1) \rangle \langle \mathcal{O}_Y(\tau_2) \rangle \right),$$

Computing the two-point function $\langle 0 | O_X(\tau_1) O_Y(\tau_2) | 0 \rangle$ in Wick-rotated time we find

$$< 0 | \mathcal{O}_X(\tau_1) \mathcal{O}_Y(\tau_2) | 0 >= -\frac{YZ}{2\Omega^2}\left(1 + 2e^{-4\Omega(\tau_2-\tau_1)}\right) + i\frac{Z}{\Omega} e^{-4\Omega(\tau_2-\tau_1)},$$

while for $\langle 0 | O_Y(\tau_1) O_X(\tau_2) | 0 \rangle$ we get

$$< 0 | \mathcal{O}_Y(\tau_1) \mathcal{O}_X(\tau_2) | 0 >= -\frac{YZ}{2\Omega^2}\left(1 + 2e^{-4\Omega(\tau_2-\tau_1)}\right) - i\frac{Z}{\Omega} e^{-4\Omega(\tau_2-\tau_1)},$$

Substituting these results we have

$$g_{XY} = \int_{-\infty}^{0} d\tau_1 \int_{0}^{\infty} d\tau_2 \left(-\frac{YZ}{2\Omega^2}\left(1+2e^{-4\Omega(\tau_2-\tau_1)}\right) + \frac{YZ}{2\Omega^2} \right),$$

where we notice that the divergent terms in this integral cancel each other, a feature that will be common in all of our examples. After integration we finally obtain

$$g_{XY} = -\frac{YZ}{16\Omega^4}.$$

Quantum Information Metric Component g_{YZ}. In this particular case we need to compute

$$g_{YZ} = \int_{-\infty}^{0} d\tau_1 \int_{0}^{\infty} d\tau_2 \left(\frac{1}{2}\langle\{\mathcal{O}_Y(\tau_1),\mathcal{O}_Z(\tau_2)\}\rangle - \langle\mathcal{O}_Y(\tau_1)\rangle\langle\mathcal{O}_Z(\tau_2)\rangle \right),$$

We have

$$< 0|\mathcal{O}_Y(\tau_1)\mathcal{O}_Z(\tau_2)|0 >= -\frac{XY}{2\Omega^2}\left(1+2e^{-4\Omega(\tau_2-\tau_1)}\right)+i\left(\frac{X}{\Omega}\right)e^{-4\Omega(\tau_2-\tau_1)},$$

$$< 0|\mathcal{O}_Z(\tau_1)\mathcal{O}_Y(\tau_2)|0 >= -\frac{XY}{2\Omega^2}\left(1+2e^{-4\Omega(\tau_2-\tau_1)}\right)-i\left(\frac{X}{\Omega}\right)e^{-4\Omega(\tau_2-\tau_1)},$$

For the product of expectation values we obtain

$$< 0|\mathcal{O}_Y(\tau_1)|0 >< 0|\mathcal{O}_Z(\tau_2)|0 >= -\frac{XY}{2\Omega^2}.$$

Substituting these results and integrating we finally obtain

$$g_{YZ} = -\frac{XY}{16\Omega^4}.$$

Quantum Information Metric Component g_{ZX}

$$g_{ZX} = \int_{-\infty}^{0} d\tau_1 \int_{0}^{\infty} d\tau_2 \left(\frac{1}{2} \langle \{ \mathcal{O}_Z(\tau_1), \mathcal{O}_X(\tau_2) \} \rangle - \langle \mathcal{O}_Z(\tau_1) \rangle \langle \mathcal{O}_X(\tau_2) \rangle \right),$$

we finally obtain

$$g_{ZX} = \frac{2Y^2 - XZ}{32\Omega^4}.$$

Quantum Information Metric Component g_{XX}

$$g_{XX} = \int_{-\infty}^{0} d\tau_1 \int_{0}^{\infty} d\tau_2 \left(\langle \mathcal{O}_X(\tau_1) \mathcal{O}_X(\tau_2) \rangle - \langle \mathcal{O}_X(\tau_1) \rangle \langle \mathcal{O}_X(\tau_2) \rangle \right),$$

where

$$< 0 | \mathcal{O}_X(\tau_1) \mathcal{O}_X(\tau_2) | 0 >= \frac{Z^2}{4\Omega^2} \left(1 + 2e^{-4\Omega(\tau_2 - \tau_1)} \right), \qquad < 0 | \mathcal{O}_X(\tau_1) | 0 > \ < 0 | \mathcal{O}_X(\tau_2) | 0 >= \frac{Z^2}{4\Omega^2},$$

Finally we get

$$g_{XX} = \frac{Z^2}{32\Omega^4}.$$

In similar form we obtain

$$g_{YY} = \frac{XZ}{8\Omega^4}, \qquad g_{ZZ} = \frac{X^2}{32\Omega^4}.$$

Having calculated all the quantum information matrix elements above, we can finally write

$$g_{ab} = \frac{1}{32\Omega^4}\begin{pmatrix} Z^2 & -2ZY & 2Y^2 - XZ \\ -2ZY & 4XZ & -2XY \\ 2Y^2 - XZ & -2XY & X^2 \end{pmatrix},$$

which is exactly the same result one obtains by direct computation.

Berry Curvature of the Generalized Harmonic Oscillator from the Lagrangian Formalism

In this section we will describe the computation of the elements of the Berry curvature following the Lagrangian formalism.

We will compute the component F_{XY} , which according to the Lagrangian formalism is given by

$$F_{XY} = \frac{1}{i}\int_{-\infty}^{0} d\tau_1 \int_{0}^{\infty} d\tau_2 \left\langle \left[\mathcal{O}_X(\tau_1), \mathcal{O}_Y(\tau_2) \right] \right\rangle,$$

Using the expressions for the correlation functions we get

$$\frac{1}{i}\Big(\langle 0 | \mathcal{O}_X(\tau_1)\mathcal{O}_Y(\tau_2) | 0 \rangle - \langle 0 | \mathcal{O}_Y(\tau_1)\mathcal{O}_X(\tau_2) | 0 \rangle \Big) = \frac{2Z}{\Omega} e^{-4\Omega(\tau_2 - \tau_1)} .$$

By integrating, we get

$$F_{XY} = \frac{Z}{8\Omega^3}.$$

Berry Curvature Component F_{ZX} , in this case we must calculate

$$F_{ZX} = \frac{1}{i}\int_{-\infty}^{0} d\tau_1 \int_{0}^{\infty} d\tau_2 \left\langle \left[\mathcal{O}_Z(\tau_1), \mathcal{O}_X(\tau_2) \right] \right\rangle,$$

with

$$\frac{1}{i}\Big(\langle 0 | \mathcal{O}_Z(\tau_1) \mathcal{O}_X(\tau_2) | 0 \rangle - \langle 0 | \mathcal{O}_X(\tau_1) \mathcal{O}_Z(\tau_2) | 0 \rangle \Big) = \frac{2Y}{\Omega} e^{-4\Omega(\tau_2 - \tau_1)}.$$

Then, we obtain

$$F_{ZX} = \frac{Y}{8\Omega^3}.$$

Berry Curvature Component F_{YZ}

From

$$F_{YZ} = \frac{1}{i}\int_{-\infty}^{0} d\tau_1 \int_{0}^{\infty} d\tau_2 \left\langle \left[\mathcal{O}_Y(\tau_1), \mathcal{O}_Z(\tau_2) \right] \right\rangle,$$

we obtain

$$F_{YZ} = \frac{X}{8\Omega^3}.$$

Therefore, the writing the Berry curvature obtained in this manner as a two-form is

$$F = \frac{1}{16\Omega^3}\left(X\, dY \wedge dZ + Y\, dZ \wedge dX + Z\, dX \wedge dY \right)$$

Conclusions

- In this talk we have taken a path-integral approach for computing the quantum information metric of a quantum field theory originally developed in the context of the gauge/gravity duality, and generalized it in order to be able to compute the Berry curvature as well in a unified manner.

- More concretely, in the generalization presented in this paper we are able to first obtain a general expression for the quantum geometric tensor, and from this we are able to compute the quantum information metric and Berry curvature of the system. This is done for any number of dimensions of the parameter-space and for the general case of non-zero expectation values for the deformation operators.

- We can see that the advantages of this generalized approach are manifold. It allows us to compute both the quantum information metric and the Berry curvature starting from a single object, the quantum geometric tensor.

- Moreover, since this is accomplished in a path-integral approach, it can also be applied in a more natural and straightforward manner in a quantum field theory setting. We should also mention that, from an operational point of view, this procedure allows to compute the information metric and the Berry curvature from expectation values and two-point functions with respect to the unperturbed original system, and could therefore be applied even in the case of systems where the exact solution is not known.