Entanglement of the gluonic field induced by a quark

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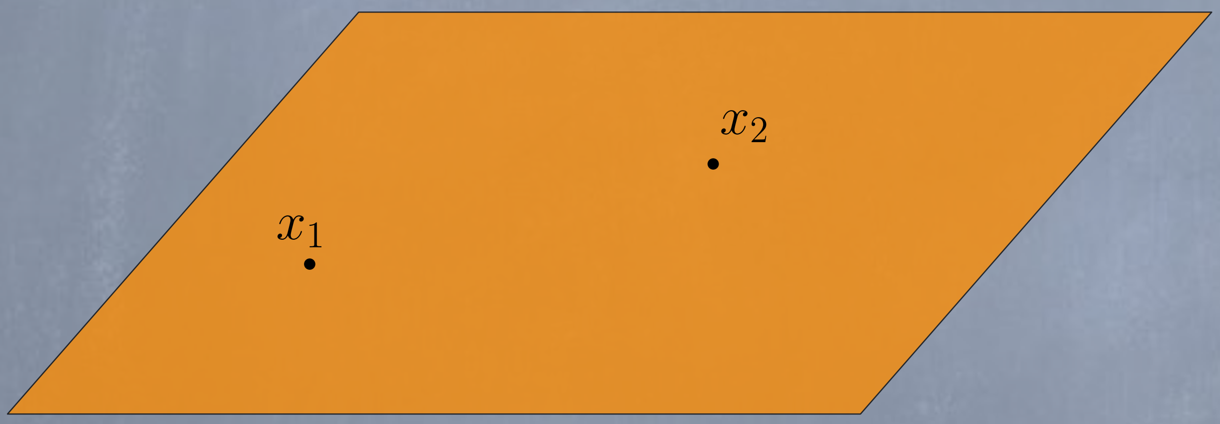
Motivation

Why Entanglement?

In a QFT we are interested in calculating correlators

$$\mathbb{G}_2 = \langle 0 | \mathcal{O}(x_1)\mathcal{O}(x_2) | 0 \rangle$$



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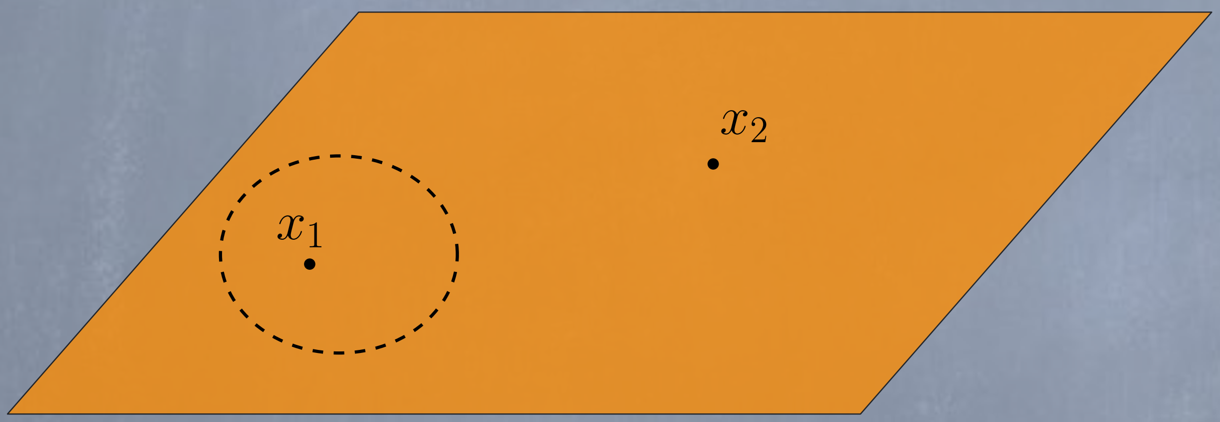

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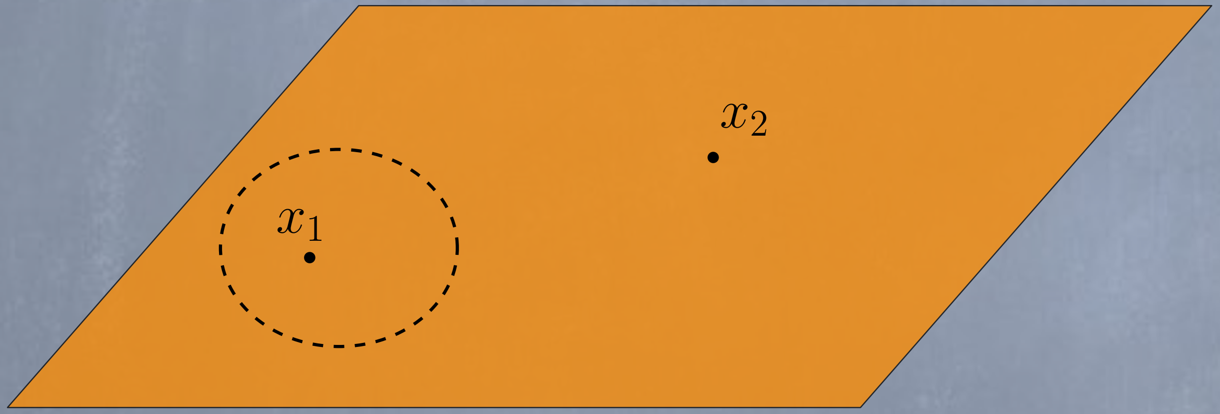

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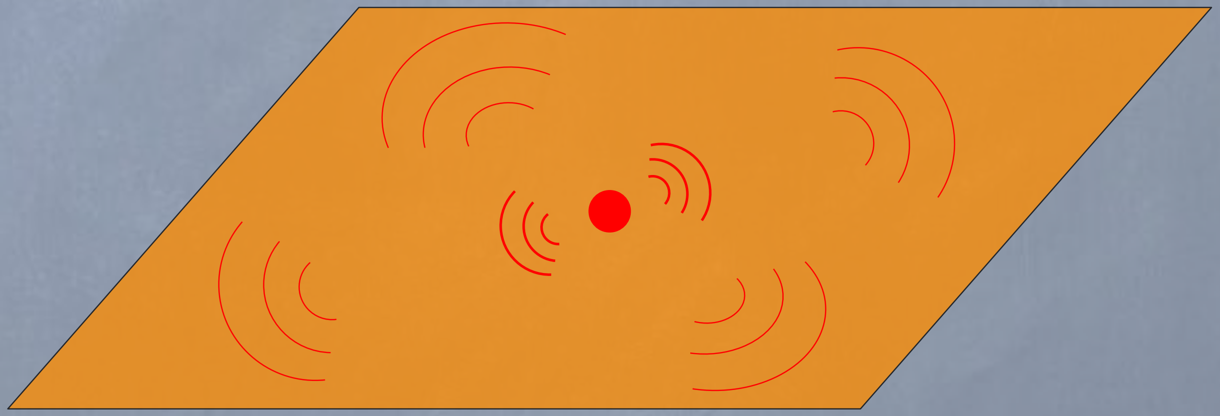
We can explore the dynamics of any gauge theory by coupling it to external charges



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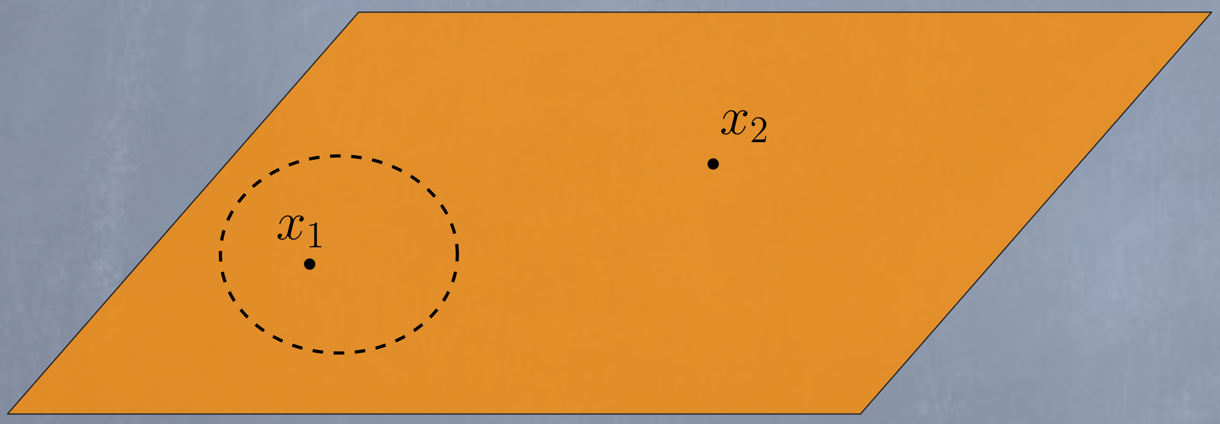

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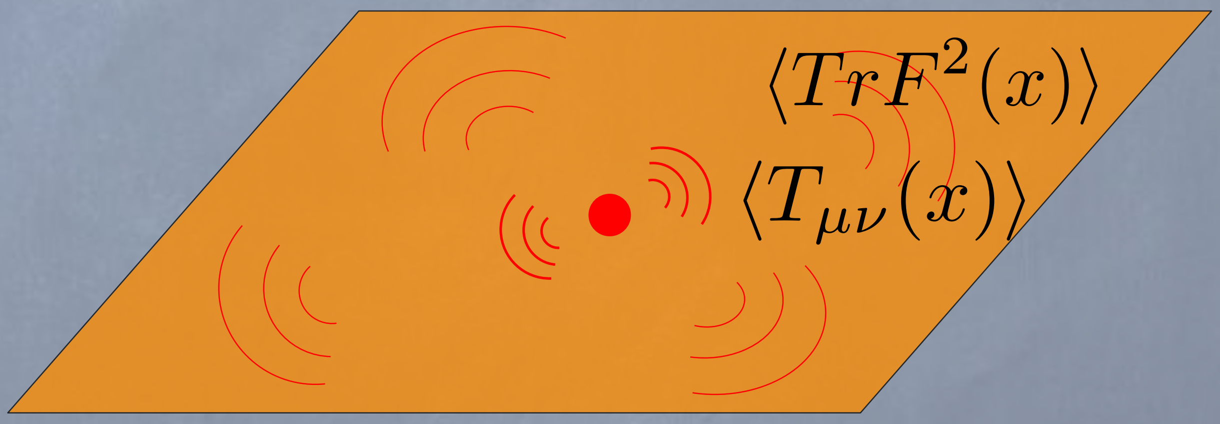

If we want to know how the charge excites the fields, we obtain this information with correlators

Or, we can calculate the entanglement pattern

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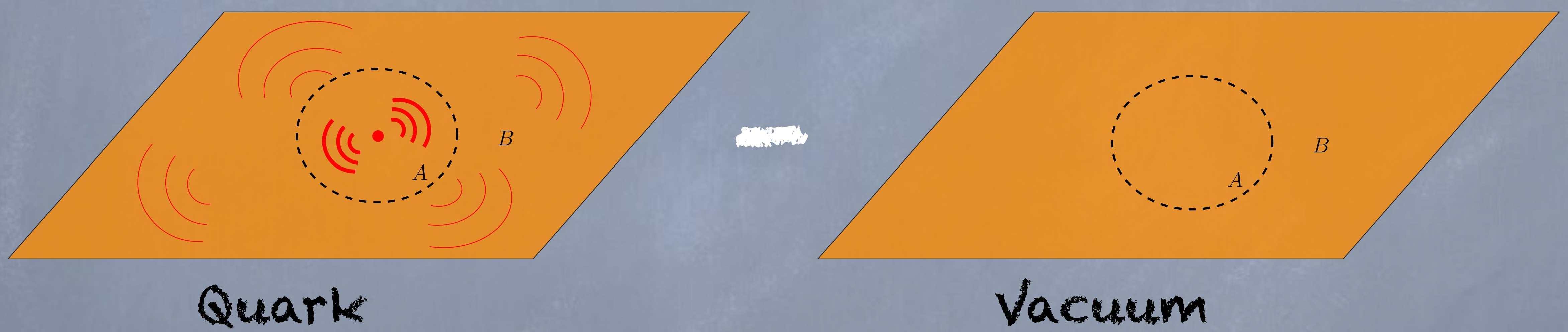

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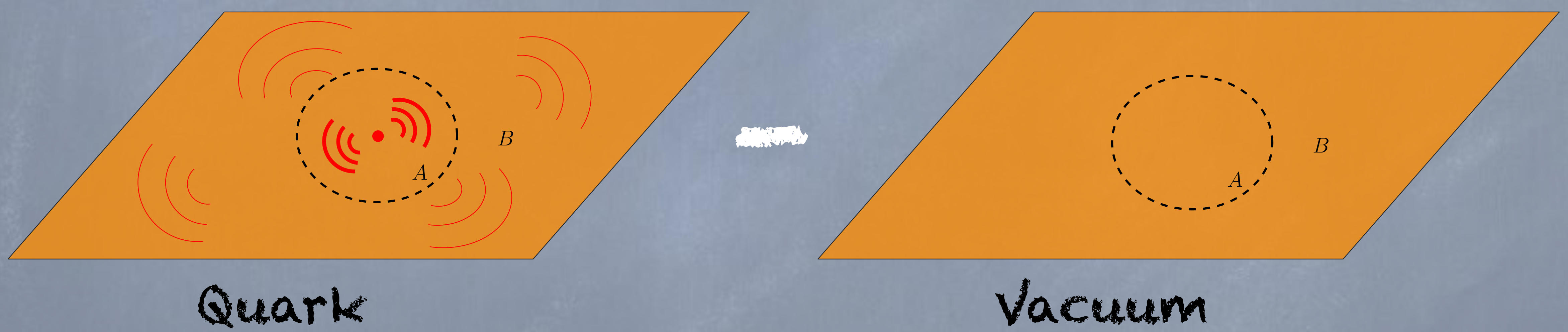


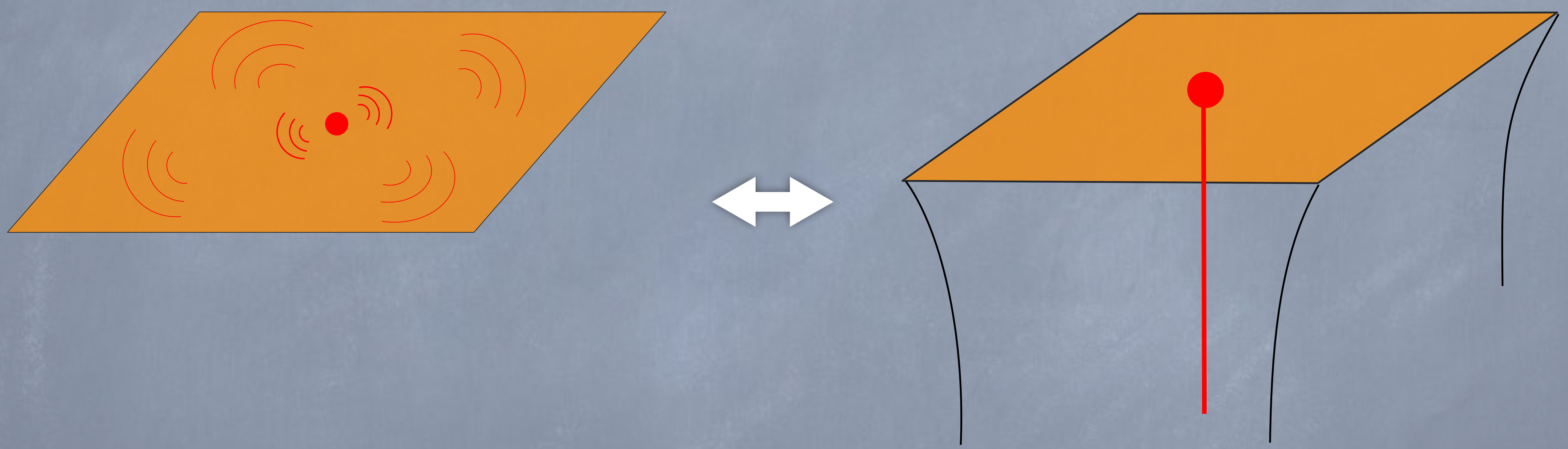
- Lewkowycz and Maldacena got the exact result for the entanglement entropy, valid for all N and λ, using localization
- We use geometric tools of the AdS/CFT correspondence

Outline

- External quarks in the AdS/CFT correspondence
- Ryu-Takayanagi prescription
- Karch-Chang's double integral formula
- The string as a probe
- Result
- Conclusions

Quarks in the AdS/CFT correspondence

- In the collection of *N* D3-branes, which give rise to a non-abelian theory, we want to add degrees of freedom which transform in the fundamental representation of the group $U(N)$



Isolated quark with mass

$m \to \infty$

Macroscopic string with one end point at

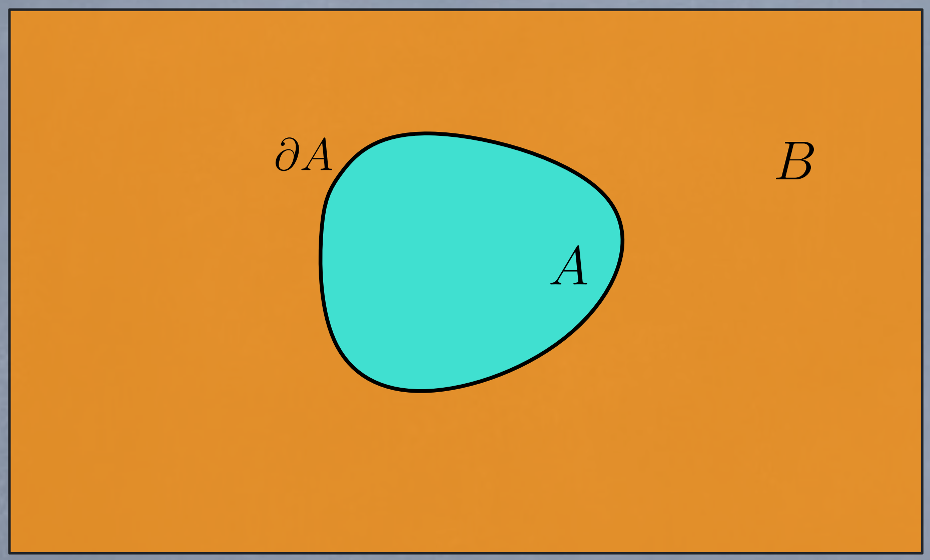
$z = 0 \quad \text{and} \quad z \to \infty$

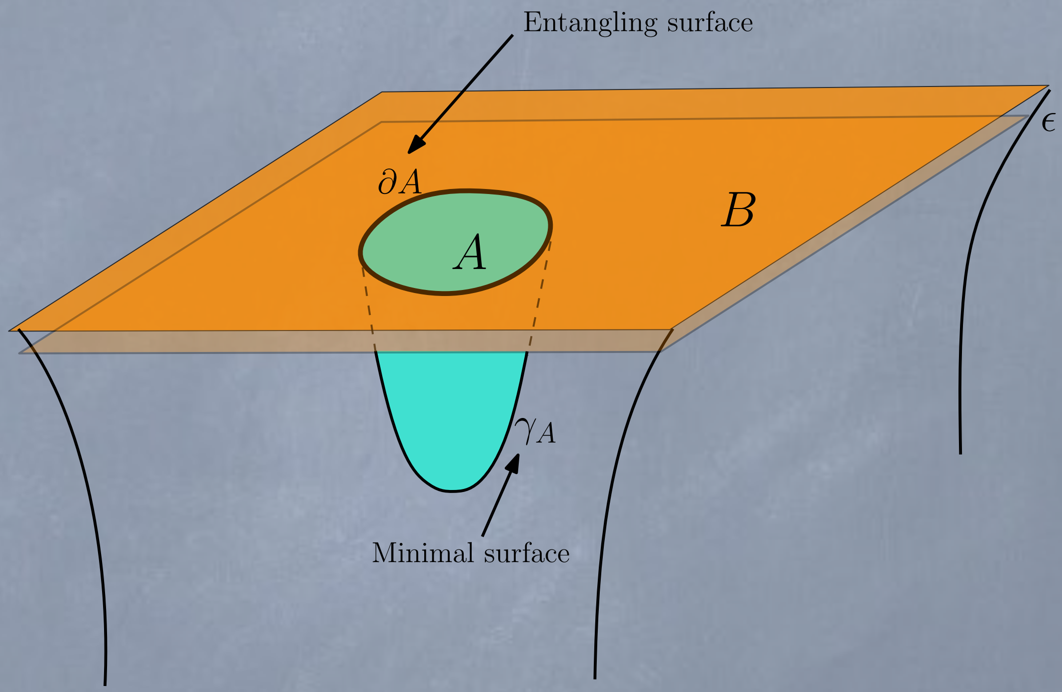
- String tip at $z = 0$ corresponds to the quark
- String body encodes the gluonic field profile induced by the quark

Ryu-Takayanagi prescription

[Ryu-Takayanagi, 2006]

- Ryu and Takayanagi established a prescription to calculate the entanglement entropy of a CFT in a time-independent state, using only geometric tools provided by AdS/CFT correspondence





$$S_A = -Tr_A(\rho_A \log \rho_A)$$

The entangling surface ∂A can be extended inside the bulk with the surface γ_A, in such a way that $\partial\gamma_A = \partial A$ is fulfilled and whose area is minimal

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+1)}}$$

Karch-Chang's double integral formula

[Karch y Chang, 2014]

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- Due to the presence of a source of matter, $T^{p'q'}_{Brane}(x')$, a fluctuation, $h_{mn} = \delta G_{mn}$, is produced around the AdS background

$$h_{mn}(x) = \kappa^2_{d+1} \int d^{d+1}x' \sqrt{G} \quad \mathcal{G}_{mn;p'q'}(x,x') T^{p'q'}_{Brana}(x')$$

[D'Hoker, Freedman, Mathus, Matusis y Rastelli, 1999]

- The coordinates x' are in the probe worldvolume

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The action for a minimal surface

$$S_{Min} = \frac{1}{4G_N^{(d+1)}} \int d^{d-1}x \sqrt{\gamma}$$

$$\gamma_{ab} = \partial_a X^m \partial_b X^n G_{mn}$$

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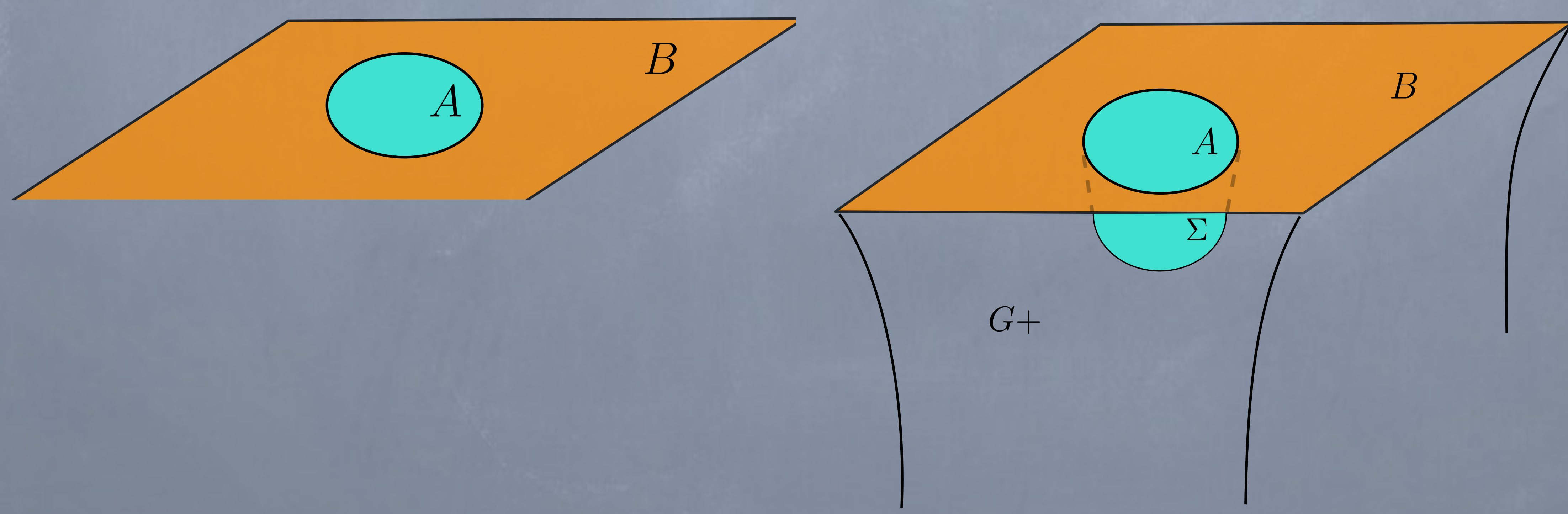
The string as a probe

We calculate the first correction induced by a quark, beyond the entanglement of the vacuum

PROBE → STRING

- We choose a spherical entangling surface, whose RT surface is a spherical cap



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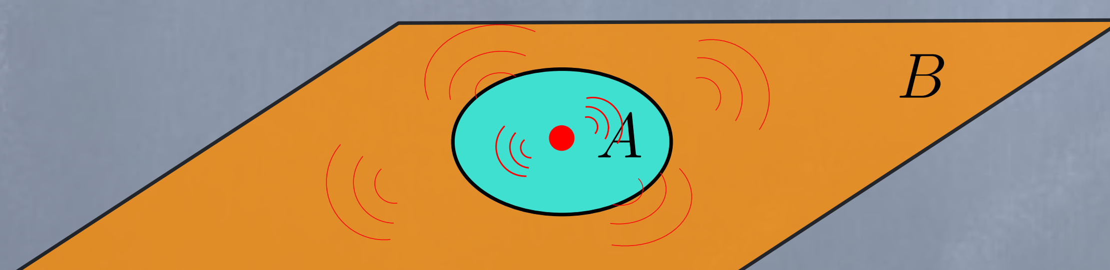
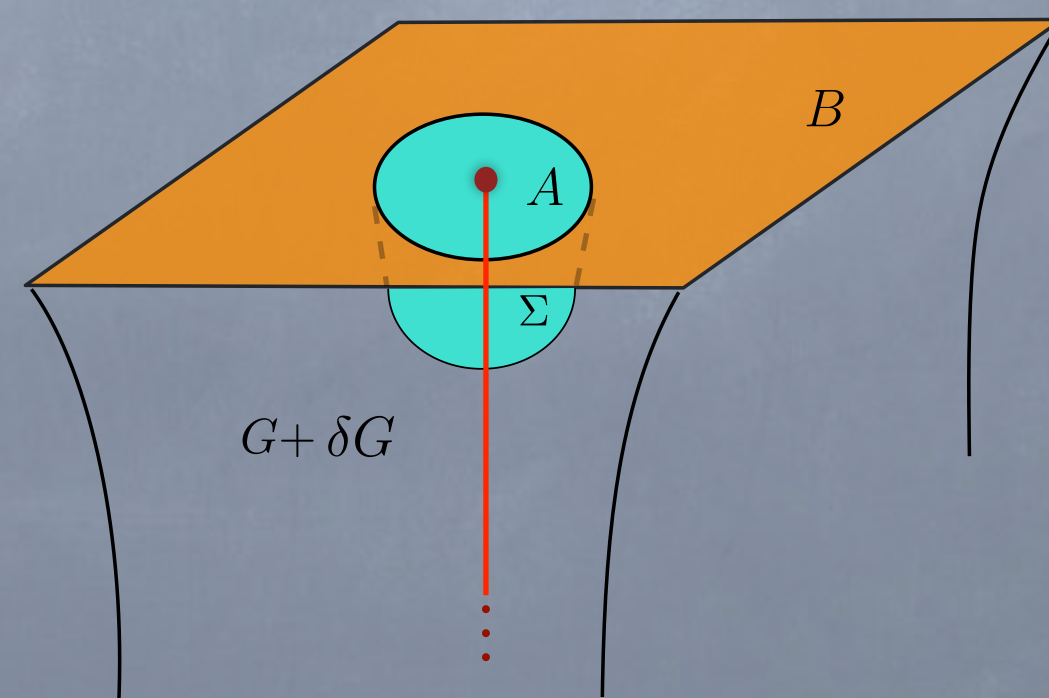
- We choose a spherical entangling surface, whose RT surface is a spherical cap
- Consider the quark at the origin of field theory, which corresponds to a vertical macroscopic string hanging down from $z = 0$
- The AdS metric, G , is modified by the string to $G' = G + \delta G$





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Energy-momentum tensor of the string

$$S_{NG}^{(\mathcal{E})} = T \int dt dz \sqrt{det(g_{\alpha\beta})}$$

Using the static gauge, the static string embedding, and varying the action, we obtain the only non-zero components of stress tensor

$$T^{tt} = T^{zz} = \frac{1}{2\pi\alpha'} \frac{z^5}{L^5} \delta^3(\vec{x})$$

hence

$$h_{mn}(z,x) = \frac{\kappa_5^2}{2\pi\alpha'} \int dz' dt' \left[\mathcal{G}_{mn;z'z'} + \mathcal{G}_{mn;t't'} \right]$$

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Euclidean propagator

They express the propagator as a linear combination of five independent bi-tensors, but we are only interested in the components with physical content

$$\mathcal{G}_{mn;m'n'}(u) = T^{(1)}_{mn;m'n'} \mathbb{H}(u) + T^{(2)}_{mn;m'n'} \mathbb{G}(u) + \ldots$$

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We obtain the perturbation up to an integral in z'

$$\boxed{\begin{gathered} h_{zt} = h_{tj} = 0 \\ h_{mn} = \frac{\kappa_5^2 L}{2\pi^3 \alpha'} \int_0^\infty dz' \left[\mathrm{Pol}^{(1)}_{mn} E\left(\frac{2\sqrt{zz'}}{\sqrt{x^2 + (z+z')^2}} \right) + \mathrm{Pol}^{(2)}_{mn} K\left(\frac{2\sqrt{zz'}}{\sqrt{x^2 + (z+z')^2}} \right) \right] \end{gathered}}$$

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Minimal "energy-momentum tensor"

By the presence of the string in the background

$$T^{mn}_{Min} \equiv \frac{2}{\sqrt{\gamma}}\frac{\delta S_{Min}}{\delta G_{mn}} = \gamma^{ab}\partial_a X^m \partial_b X^n$$

where $\gamma_{ab} = \partial_a X^m \partial_b X^n G_{mn}$

Taking advantage of the spherical symmetry of the configuration and choosing $X^a = \{z, \theta, \phi\}$ as the coordinates over the cap

$$T^{zz}_{Min} = \frac{z^2}{L^2}\frac{R^2 - z^2}{R^2} \qquad T^{rr}_{Min} = \frac{z^4}{L^2 R^2}$$

$$T^{zr}_{Min} = -\frac{z^3\sqrt{R^2 - z^2}}{L^2 R^2} \qquad T^{tt}_{Min} = 0$$

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The perturbation in spherical coordinates

$$h_{zt} = h_{z\theta} = h_{z\phi} = h_{tr} = h_{t\theta} = h_{t\phi} = h_{r\theta} = h_{r\phi} = h_{\theta\phi} = 0$$

$$h_{mn} = \frac{\kappa_5^2 L}{2\pi^3 \alpha'} \int_0^\infty dz' \Bigg[\mathrm{Pols}^{(1)}_{mn}(z; z') E\left(\frac{2\sqrt{zz'}}{\sqrt{R^2 + 2zz' + z'^2}} \right) + \mathrm{Pols}^{(2)}_{mn}(z; z') K\left(\frac{2\sqrt{zz'}}{\sqrt{R^2 + 2zz' + z'^2}} \right) \Bigg]$$

Doing the convolution and integrating

$$\delta S_{Min} = \frac{1}{8G_N^{(5)}} \frac{L^2}{\alpha'} \int_0^\infty dz' \int_\epsilon^R dz \frac{\kappa_5^2 \sqrt{R^2 - z^2}}{15\pi^2 R z^5 z'^4 (R^2 - 2zz' + z'^2) \sqrt{R^2 + 2zz' + z'^2}} \left[\mathrm{Pol}^{(1)} E\left(\frac{2\sqrt{zz'}}{\sqrt{R^2 + 2zz' + z'^2}} \right) + \mathrm{Pol}^{(2)} K\left(\frac{2\sqrt{zz'}}{\sqrt{R^2 + 2zz' + z'^2}} \right) \right]$$

Karch-Chang's double integral formula for the string

$$\delta S_{Min} = \frac{1}{4G_N^{(5)}} \int d^3 w \frac{\sqrt{\gamma}}{2} T^{mn}_{Min} (\delta G)_{mn}$$

Minimal "energy-momentum tensor"

By the presence of the string in the background

$$T^{mn}_{Min} \equiv \frac{2}{\sqrt{\gamma}} \frac{\delta S_{Min}}{\delta G_{mn}} = \gamma^{ab} \partial_a X^m \partial_b X^n$$

where $\gamma_{ab} = \partial_a X^m \partial_b X^n G_{mn}$

Taking advantage of the spherical symmetry of the configuration and choosing $X^a = \{z, \theta, \phi\}$ as the coordinates over the cap

$$T^{zz}_{Min} = \frac{z^2}{L^2} \frac{R^2 - z^2}{R^2} \qquad T^{rr}_{Min} = \frac{z^4}{L^2 R^2}$$

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The perturbation in spherical coordinates

$$h_{zt} = h_{z\theta} = h_{z\phi} = h_{tr} = h_{t\theta} = h_{t\phi} = h_{r\theta} = h_{r\phi} = h_{\theta\phi} = 0$$

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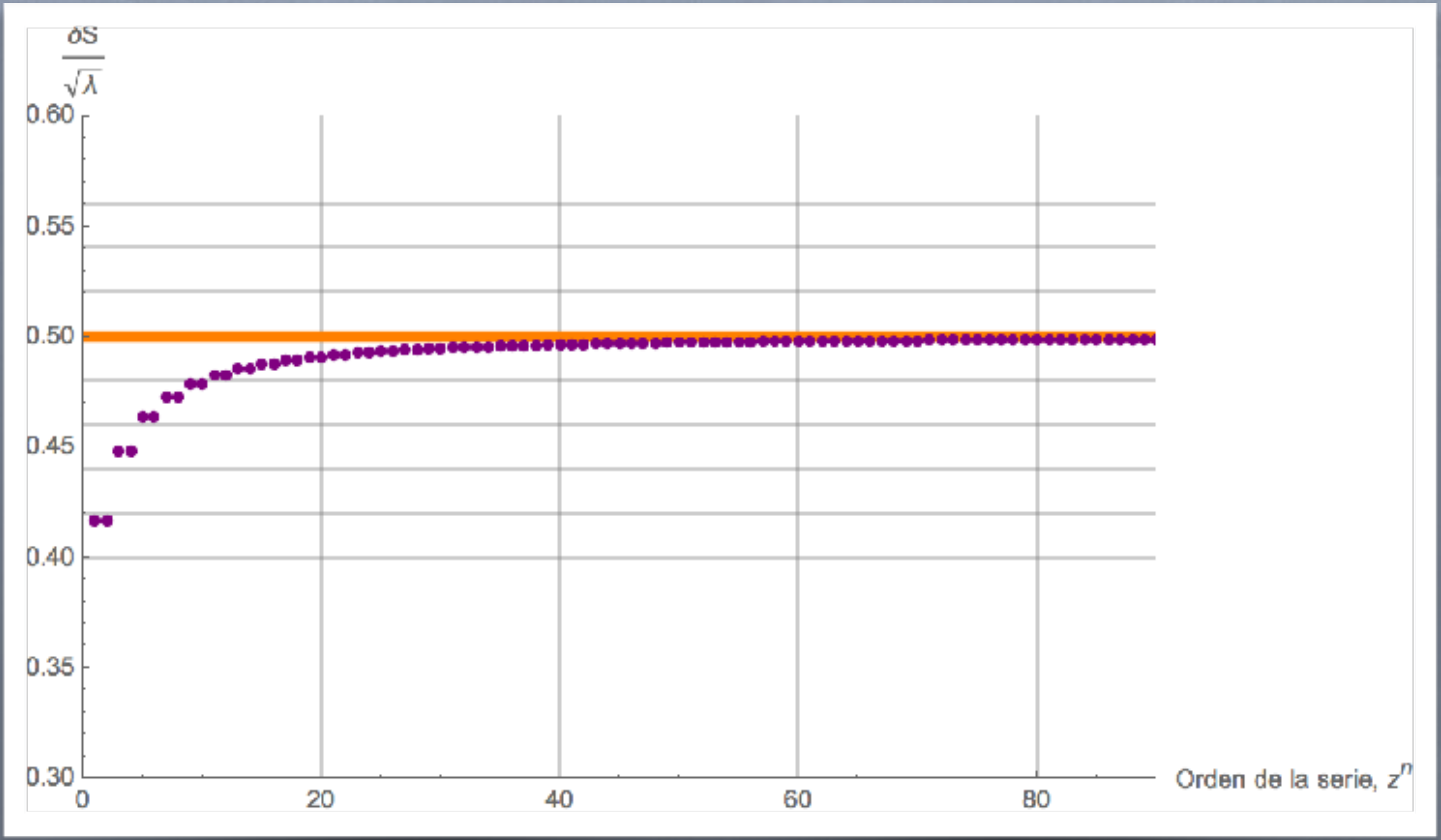
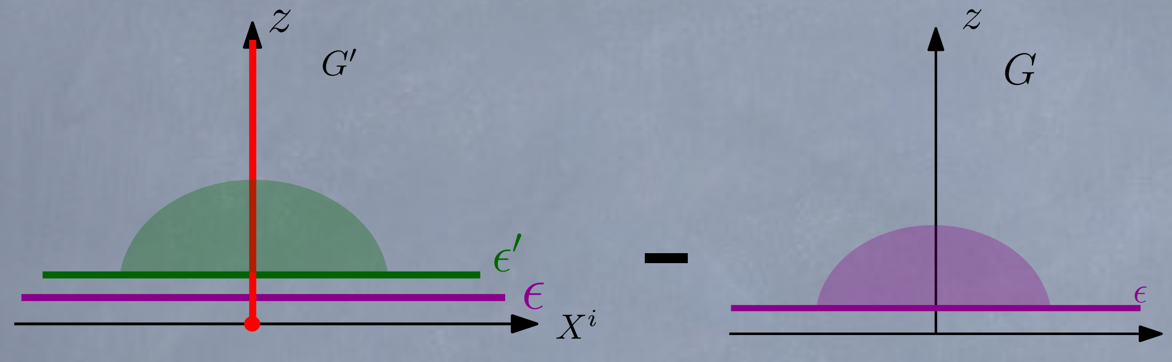
Karch and Chang's double integral formula for the string

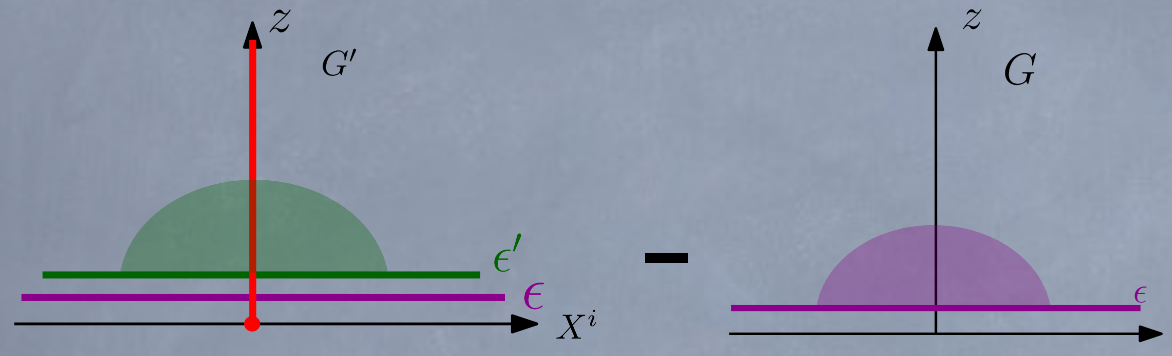
$$\delta S_{Min} = \frac{\sqrt{\lambda}}{15\pi R}\int_0^\infty dz' \int_\epsilon^R dz \frac{\sqrt{R^2-z^2}}{z^5 z'^4 (R^2 - 2zz' + z'^2)\sqrt{R^2+2zz'+z'^2}}$$
$$\left[\mathrm{Pol}^{(1)} E\left(\frac{2\sqrt{zz'}}{\sqrt{R^2+2zz'+z'^2}}\right) + \mathrm{Pol}^{(2)} K\left(\frac{2\sqrt{zz'}}{\sqrt{R^2+2zz'+z'^2}}\right)\right]$$
$$\sim \frac{\sqrt{\lambda}}{2}$$



Regularization term



Regularization term



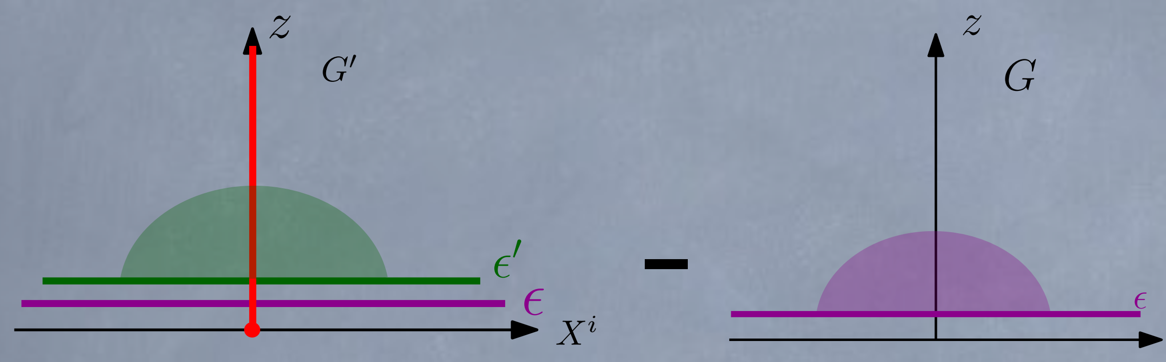
In order to have the same regularization process on both sides

$$\det(\{\Gamma_\epsilon\}_{ab}) = \det(\{\Gamma_{\epsilon'}\}_{ab})$$

[Karch y Chang, 2014; Taylor y Woodhead, 2016]

where $\{\Gamma_\epsilon\}_{ab}$ is the induced metric

Regularization term



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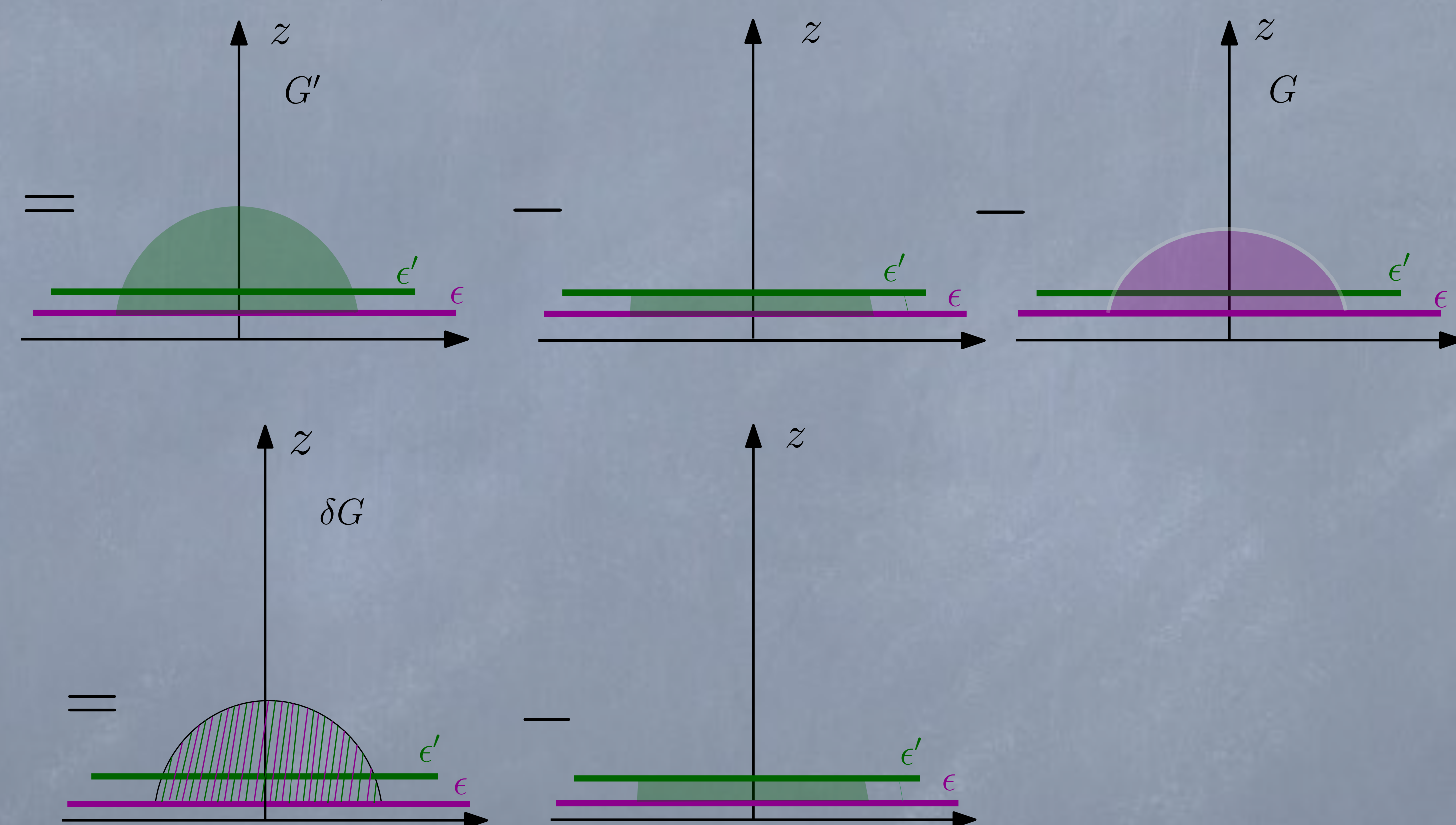

$$\delta S = \delta S_{Min} + I_R$$

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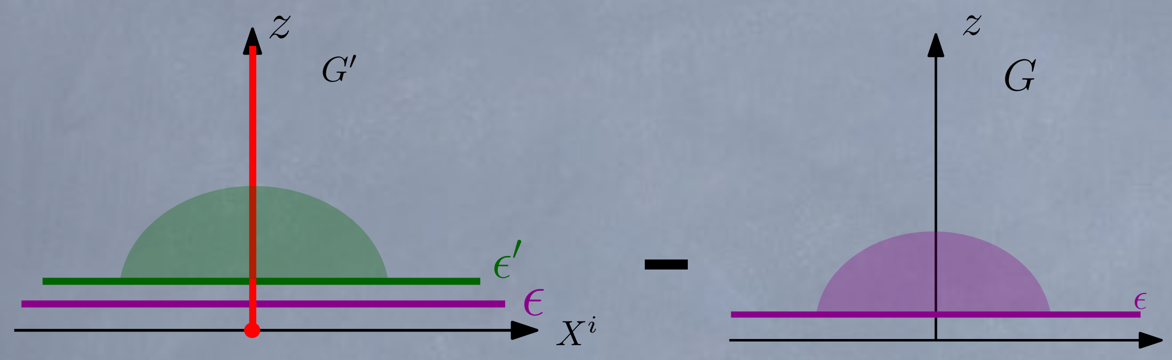

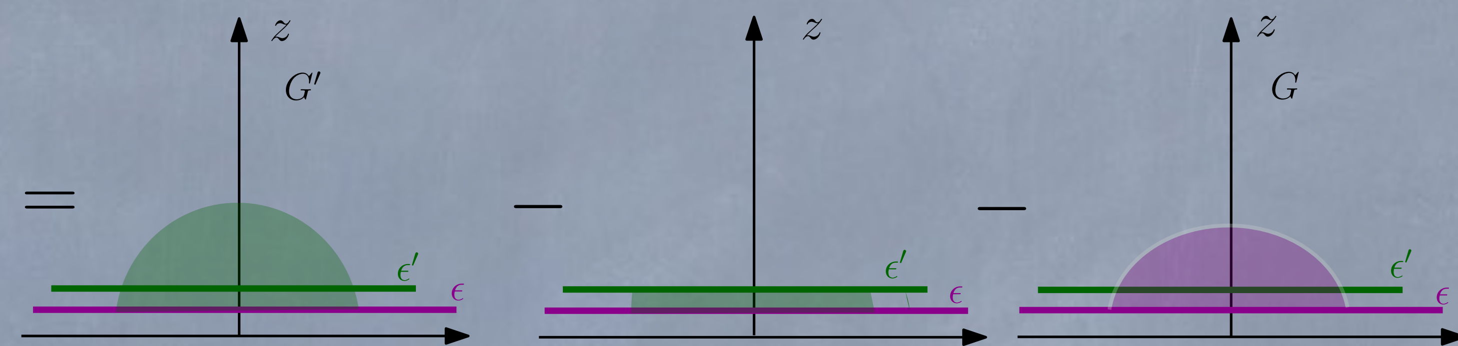

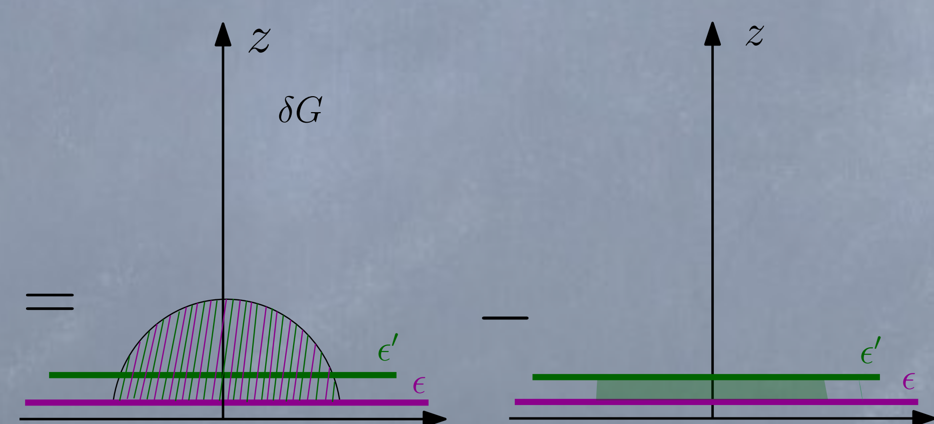

$$\delta S = \underbrace{\delta S_{Min}}_{} + I_R$$

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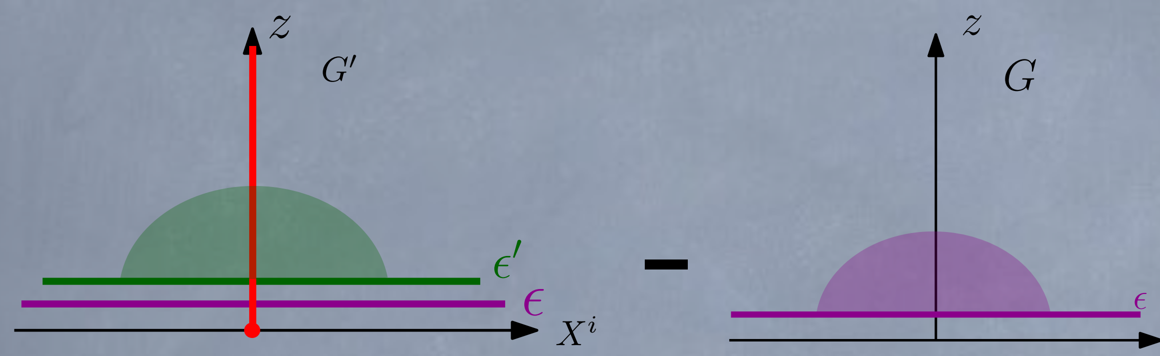
Regularization term



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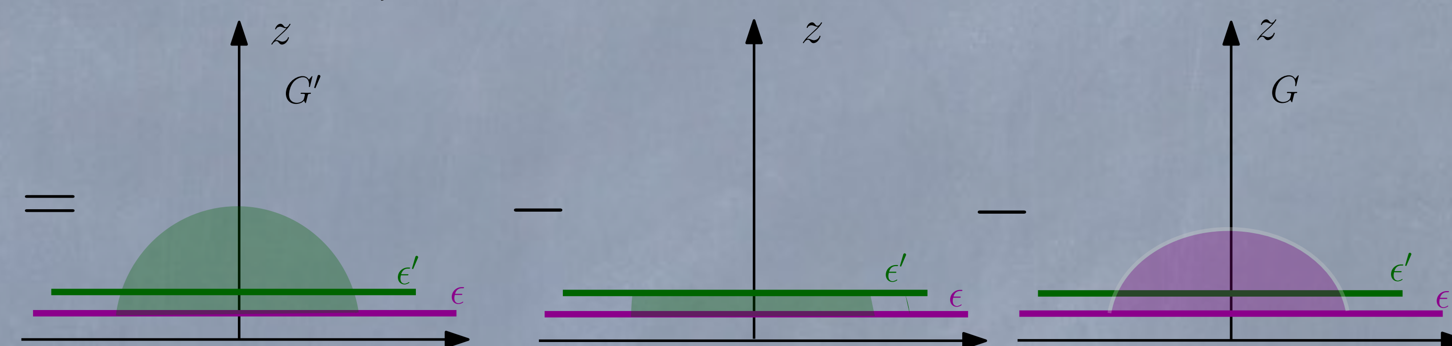

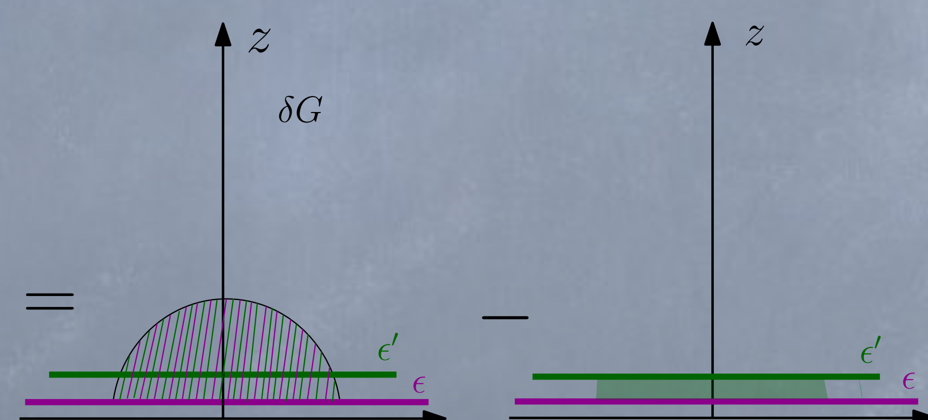

$$\delta S = \underbrace{\delta S_{Min}} + \underbrace{I_R}$$

$$= \frac{\sqrt{\lambda}}{2} \boxed{-\frac{1}{4G_N^{(5)}} \int_{\epsilon}^{\epsilon'} dz \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sqrt{\gamma}}$$

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The proposal is

$$\det(\{\Gamma_{\epsilon}\}_{ab}) = \det(\{\Gamma_{\epsilon'}\}_{ab})$$

The induced metric in ϵ'

$$\{\Gamma_{\epsilon'}\}_{ab} = \partial_a X'^m \partial_b X'^n G'_{mn} = \partial_a X'^m \partial_b X'^n (G_{mn} + h_{mn})$$

$$X'^m = (z = \epsilon' = \epsilon + \delta z, t, r = \sqrt{R^2 - (\epsilon + \delta z)^2}, \theta, \phi)$$

We obtain δG_{mn} considering that we are near to the boundary, $z \to 0$. Thereby the only non-zero components of the induced metric to first order in

$$\Gamma'_{tt} \qquad \Gamma'_{rr} \qquad \Gamma'_{\theta\theta} \qquad \Gamma'_{\phi\phi}$$

The induced metric in ϵ

$$\Gamma_{tt} = \frac{L^2}{\epsilon^2}, \qquad \Gamma_{rr} = \frac{L^2}{\epsilon^2},$$

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Match the determinants and solve

$$\delta z = \frac{G_N}{6\pi\alpha' L R^2}\epsilon^3 + \mathcal{O}(\epsilon^5)$$

$$I_R = -\frac{L^2}{6\alpha'} = -\frac{\sqrt{\lambda}}{6}$$

Regularization term

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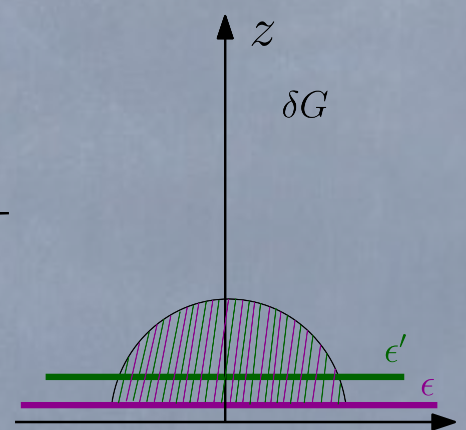
Result

The change in entanglement entropy to first order due to the string is

$$\delta S = \delta S_{Min} + I_R$$

$$= \frac{1}{4G_N^{(5)}} \Bigg[$$



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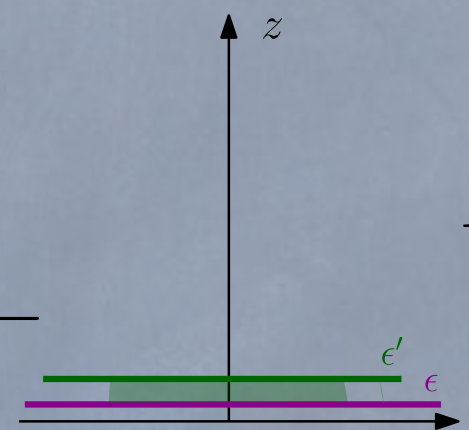

$$\Bigg]$$

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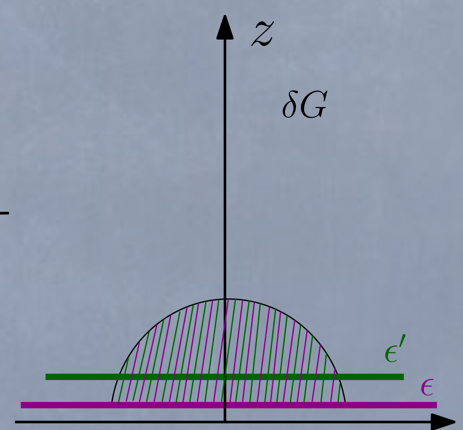

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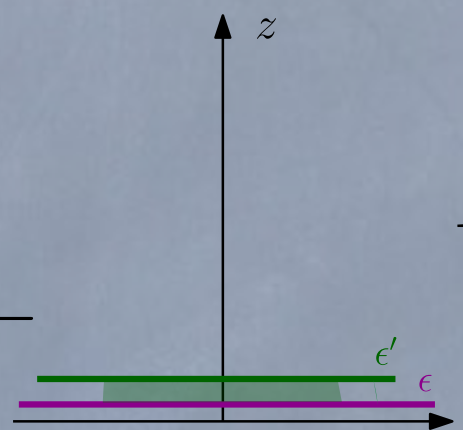

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Lewkowycz and Maldacena's result

$$S \approx_{N\to\infty} \log\left(\frac{2N}{\sqrt{\lambda}} I_1(\sqrt{\lambda})\right) - \frac{2}{3}\frac{\sqrt{\lambda} I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})}$$

So

$$S(\lambda >> 1) \approx \frac{\sqrt{\lambda}}{3} - \frac{3}{4}\log\lambda + \ldots$$

Conclusions and perspectives

- With Karch-Chang's double integral formula we approached to the result $\sqrt{\lambda}/2$. Including the necessary regularization term, which contributes with $-\sqrt{\lambda}/6$, our final result is $\delta S = \sqrt{\lambda}/3$
- The result does not depend on the radius of the entanglement surface. This is expected by conformal invariance
- The usual area law term is canceled between S_{quark} and S_{vacuum}
- Our result is completely in agreement with the limit $N \to \infty, \quad \lambda \to \infty$ of the result reported by Lewkowycz and Maldacena
- Our result tests the validity of Karch-Chang's formula, and it also tests the calculation technique provided by Lewkowycz and Maldacena

- We intend to extend this for a static quark with finite mass.
- It would be interesting (but difficult) to explore the entanglement pattern for a quark in motion