Resonant dynamics and the (in)stability of AdS

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Based on:

arXiv:1612.04758, F. Dimitrakopoulos, B. Freivogel & JFP
arXiv:1607.08094, F. Dimitrakopoulos, B. Freivogel, JFP & I. Yang

Outline

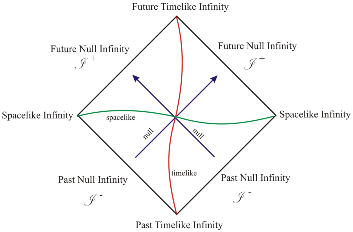
Motivation

Main question

Is AdS stable under arbitrarily small perturbations?

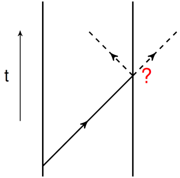
i) Interesting from mathematical relativity perspective
Minkowski space is stable due to dispersion of energy to infinity [Christodoulou & Klainerman]:

- For sufficiently large amplitudes: black hole formation
- For sufficiently small amplitudes: shell scatters back to infinity

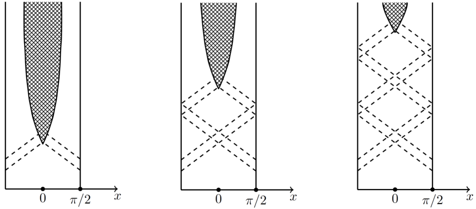


Motivation

AdS is not globally hyperbolic, acts as a box:



Boundary reflects the shell and it has infinitely many chances to collapse



Stability is difficult to prove!

Motivation

ii) Interesting from AdS/CFT perspective.

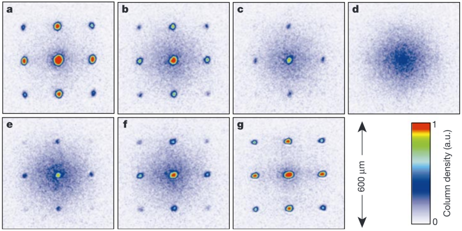
- Black hole formation = thermalization of CFT
- Non-collapsing solutions = quantum revivals?

letters to nature

Collapse and revival of the matter wave field of a Bose–Einstein condensate

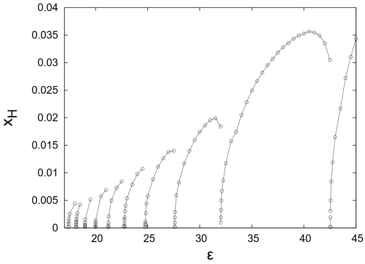
Markus Greiner, Olaf Mandel, Theodor W. Hänsch & Immanuel Bloch

Sektion Physik, Ludwig-Maximilians-Universität, Schellingstrasse 4/III, D-80799 Munich, Germany, and Max-Planck-Institut für Quantenoptik, D-85748 Garching, Germany



Review of previous works

- [Bizon & Rostworowski] considered a massless scalar field in AdS. Found collapses at small (but finite) ε:

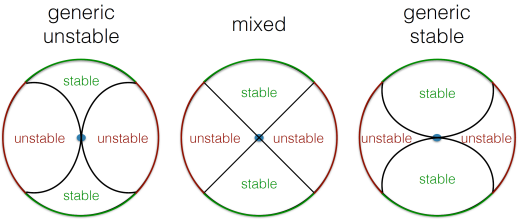


  - Choptuik scaling $x_H \sim (\varepsilon - \varepsilon_*)^\gamma$, with $\gamma \approx 0.37$ [Choptuik]
  - Extrapolation to $\varepsilon \to 0$ limit requires high computational cost

- [Dias, Horowitz, Marolf & Santos] and [Balasubramanian, Buchel, Green, Lehner & Liebling] found families of initial condition that do not lead to collapse, conjectured to be of measure zero!

Review of previous works

Possible outcomes of the instability in the $\varepsilon \to 0$ limit:



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Review of previous works

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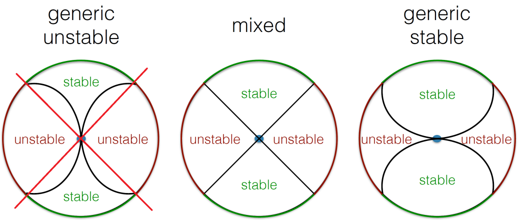

- Various works have ruled out the first option [Buchel, Green, Lehner & Liebling], [Dimitrakopoulos & Yang]
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Review of previous works

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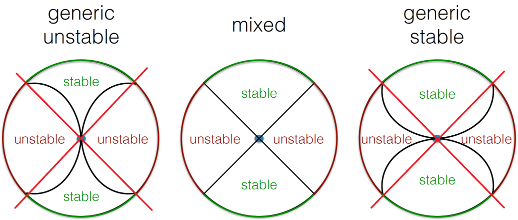


- Various works have ruled out the first option [Buchel, Green, Lehner & Liebling], [Dimitrakopoulos & Yang]
- In arXiv:1612.04758 we gave evidence in support of the mixed diagram. Very recently it was formally proved by [Moschidis].

Setup of the problem

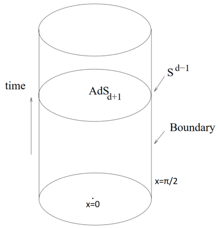
- AdS is the maximally symmetric solution of the vacuum Einstein equations $R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta} = 0$, with negative Λ:

$$ds^2 = -(1 + r^2/L^2)d\tilde{t}^2 + \frac{dr^2}{1 + r^2/L^2} + r^2 d\Omega^2$$

where $L^2 = -d(d-1)/(2\Lambda)$, $r \geq 0$ and $-\infty < t < \infty$

- Substituting $r = L \tan x$ $(0 \leq x \leq \pi/2)$, and $\tilde{t} = Lt$ we get

$$ds^2 = \frac{L^2}{\cos^2 x}\left(-dt^2 + dx^2 + \sin^2 x d\Omega^2\right)$$



Setup of the problem

- Consider a massless scalar field coupled to gravity with negative Λ:

$$S=\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R-2\Lambda-\frac{1}{2}(\partial\phi)^2\right]$$

- We assume spherical symmetry. The ansatz for the metric is:

$$ds^2=\frac{1}{\cos^2 x}\left(-Ae^{-2\delta}dt^2+A^{-1}dx^2+\sin^2 x d\Omega^2\right)$$

- Introducing $\Phi=\phi'$ and $\Pi=A^{-1}e^{\delta}\dot{\phi}$, we get the following EOM:

$$\dot{\Phi}=\left(Ae^{-\delta}\Pi\right)', \qquad \dot{\Pi}=\frac{1}{\tan^2 x}\left(\tan^2 x Ae^{-\delta}\Phi\right)'$$

and the constraints:

$$\begin{aligned}
A' &= \frac{1+2\sin^2 x}{\sin x\cos x}(1-A)-\sin x\cos x A\left(\Phi^2+\Pi^2\right)\\
\delta' &= -\sin x\cos x\left(\Phi^2+\Pi^2\right)
\end{aligned}$$

Setup of the problem

- We expand ϕ, A, δ in powers of ε:

$$\phi = \sum_{j=0}^{\infty} \varepsilon^{2j+1}\phi_{2j+1}(t,x), \quad A = 1 - \sum_{j=0}^{\infty} \varepsilon^{2j} A_{2j}(t,x),$$

$$\delta = \sum_{j=0}^{\infty} \varepsilon^{2j}\delta_{2j}(t,x),$$

  and solve the system order by order in ε:

- At order $\mathcal{O}(\varepsilon)$:

$$\ddot{\phi}_1 + \mathcal{L}\phi_1 = 0\,, \qquad \mathcal{L} \equiv -\frac{1}{\tan^2 x}\partial_x\left(\tan^2 x\partial_x\right)$$

  with eigenvalues $\omega_j = 2j+3$ and general solution:

$$\phi_1(t,x) = \sum_{j=0}^{\infty}\left(\alpha_j e^{i\omega_j t} + \bar{\alpha}_j e^{-i\omega_j t}\right) e_j(x)$$

Setup of the problem

- At order $\mathcal{O}(\varepsilon^2)$:

$$
\begin{aligned}
A_2(t,x) &= -\nu(x)\int_0^x \left(\dot{\phi}_1(t,y)^2+\phi'(t,y)^2\right)\mu(y)dy \\
\delta_2(t,x) &= \begin{cases} -\int_0^x \left(\dot{\phi}_1(t,y)^2+\phi'(t,y)^2\right)\nu(y)\mu(y)dy, & \delta(t,0)=0 \\ \int_x^{\pi/2} \left(\dot{\phi}_1(t,y)^2+\phi'(t,y)^2\right)\nu(y)\mu(y)dy, & \delta(t,\pi/2)=0 \end{cases}
\end{aligned}
$$

$\rightarrow$Importantly: Evolution is independent of the gauge arXiv:1607.08094

- At order $\mathcal{O}(\varepsilon^3)$ we obtain the first non-trivial dynamics:

$$\ddot{\phi}_3+\mathcal{L}\phi_3=\mathcal{S}_3(t,x)$$

$\rightarrow$ Expand ϕ_3 in eigenfunctions, $\phi_3=\sum c_n(t)e_n(x)$:

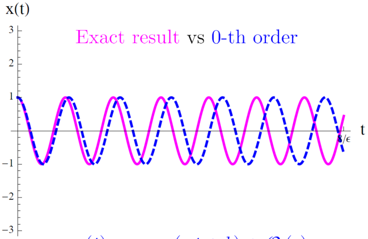
$$\ddot{c}_n+\omega_n^2 c_n=\Omega_{ijkn}a_i a_j a_k \cos[(\omega_i\pm\omega_j\pm\omega_k)t+(b_i\pm b_j\pm b_k)]$$

DANGER!! Spectrum is fully resonant $\rightarrow$ may lead to secular growths

Setup of the problem

Consider a simple example:

- Particle in potential $V(x) = \frac{\omega^2}{2}x^2 + \frac{\epsilon}{4}x^4$
- Equation of motion $\ddot{x} + \omega^2 x + \epsilon x^3 = 0$
- Perturbative expansion $x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots$



$x(t) = a\cos(\omega t + b) + \mathcal{O}(\epsilon)$

$\omega = 1,\ \epsilon = 0.2,\ a = 1,\ b = 0$

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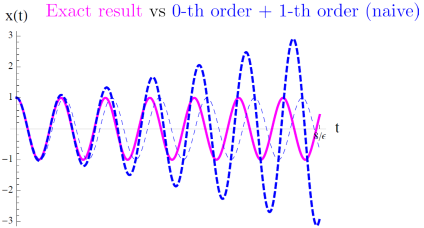

$$x(t) = a\cos(\omega t + b) + \left(\frac{a^3}{32\omega^2}\cos(3\omega t + 3b) - \frac{3a^3}{8\omega} t \sin(\omega t + b)\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

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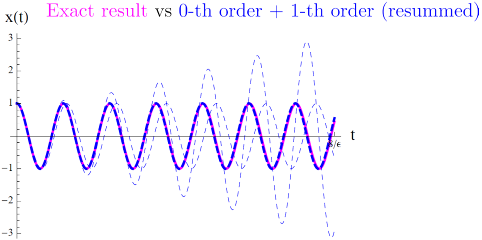


$$x(t) = a\cos\left(\left(\omega + \frac{3a^2}{8\omega^2}\epsilon\right)t + b\right) + \frac{a^3}{32\omega^2}\cos\left(3\left(\omega + \frac{3a^2}{8\omega^2}\epsilon\right)t + 3b\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

Divergence may be reabsorbed into frequency shift! [Poincaré & Lindstedt]

Setup of the problem

Method adapted to AdS dynamics is known under different names: Two Time Formalism (TTF), Resonant approximation, Time-average or Renormalization method [Balasubramanian, Buchel, Green, Lehner & Liebling], [Craps, Evnin & Vanhoof]

The idea is to consider dependence on two different time-scales, e.g. $\phi = \phi(t, \tau, x)$ where $\tau = \epsilon^2 t$ is the slow time

- At leading order one obtains

$$\phi_1(t,\tau,x) = \sum_{j=0}^{\infty} \left(\alpha_j(\tau) e^{-i\omega_j t} + \bar{\alpha}_j(\tau) e^{-i\omega_j t}\right) e_j(x)$$

- At third order one obtains:

$$\partial_t \phi_3 + L\phi_3 + 2\partial_t \partial_\tau \phi_1 = \mathcal{S}_3(\phi_1, A_2, \delta_2)$$

The extra term cancel the potential resonances if

$$\boxed{-2i\omega_j \partial_\tau \alpha_j(\tau) = \sum_{klm} S_{jklm} \bar{\alpha}_k \alpha_l \alpha_m}$$

Setup of the problem

$$\boxed{-2i\omega_j\partial_\tau\alpha_j(\tau) = \sum_{klm} S_{jklm}\bar{\alpha}_k\alpha_l\alpha_m}$$

Comments on the TTF equations:

- Scale invariant $\alpha_j(\tau) \to \varepsilon\alpha_j(\tau/\varepsilon^2)$. Crucial for extrapolation to the $\varepsilon \to 0$ limit!
- Admits two conserved quantities, energy and particle number

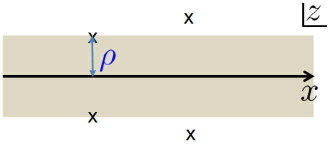
$$E = \sum_j \omega_j^2|\alpha_j|^2\,, \qquad N = \sum_j \omega_j|\alpha_j|^2\,.$$

  Implies both direct and inverse cascades. Crucial for existence of non-collapsing solutions
- Fourier asymptotics can diagnose singularity formation (turbulent instability)

Setup of the problem

Analyticity strip method

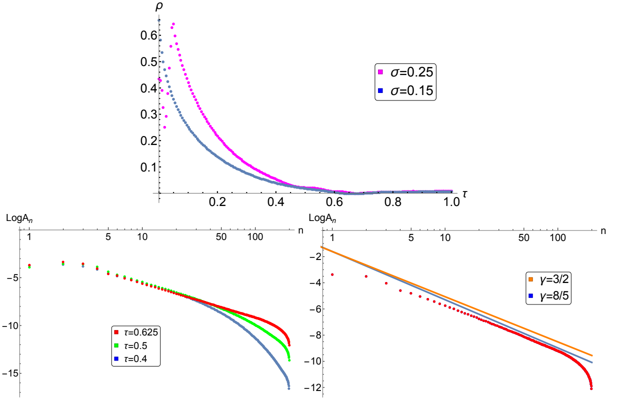
- Consider a solution $u(t, x)$ of the EOM for real-analytic initial data
- Analytic extension to complex plane $x \to z$ will generally have complex singularities moving in time
- If a complex singularity hits the real axis, $u(t, x)$ becomes singular.



- Closest pole to real axis $z = x + i\rho$; $\rho(t)$ determines width of analyticity strip
- $\rho(t)$ is encoded in $exp(-i\rho k)$ decay of Fourier coefficients of $u(t, x)$

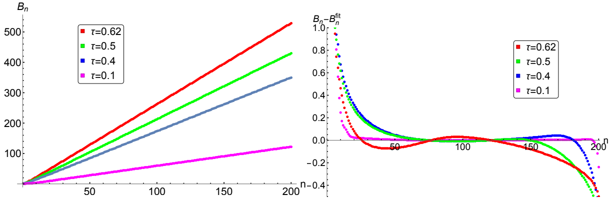
Results: Gaussian data

- Consider initial condition, $\phi(0,x) \sim \exp\left(-\frac{\tan^2 x}{\sigma^2}\right)$
- Full GR simulations lead to collapse for $\sigma < 0.3$. We evolved the system for $\sigma = 0.15$ and $\sigma = 0.25$ truncating to $n_{\max} = 200$ modes
- Amplitudes are fitted with the formula $A_n = \alpha(\tau) n^{-\gamma(\tau)} \exp(-\rho(\tau) n)$:



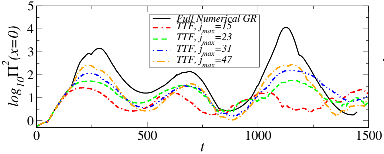
Results: Gaussian data

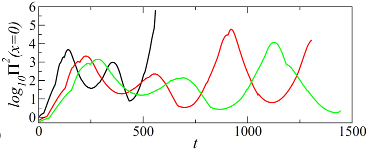
- Analyticity strip goes to zero at finite τ, as expected
- We could extract the power law behavior $A_n \sim n^{-3/2}$ predicted by [Freivogel & Yang] assuming phase coherence. This exponent differs from the $\gamma = 8/5$ found in previous literature
- Phases seem to be coherent to good extent (relative error $\sim 0.1\%$!), in stark contrast with the Kolmogorov theory of turbulence



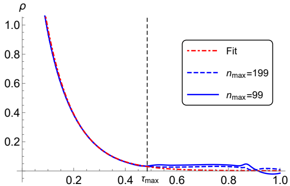
Results: Two-mode data

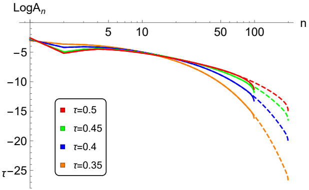
- Consider initial condition $A_n = \frac{\epsilon}{3}\left(\delta_n^0 + \kappa\delta_n^1\right)$
- This initial condition has been conjectured to be in the border of a stability island
- Conflicting GR results [Bizon et. al.] vs [Buchel et. al.]





Our results for $\kappa = 3/5$:





Results: Two-mode data

- $\rho(\tau)$ decreases and stays small but never crosses zero
- Spectrum never becomes power law during our evolution
- Extrapolation to $\tau \to \infty$ suggests that $\rho(\tau) \to 0$ at infinite slow time
- Supported by an analytic argument:
  Assuming that the preferred energy transfer channel is $0 + 1 \to n + (n+1)$ we can get an effective EOM:

$$\frac{\partial \alpha}{\partial \tau} + i d_n \alpha + c_n \frac{\partial \alpha}{\partial n} = 0$$

  This suggests a speed of propagation,

$$\text{speed} = \frac{dn}{d\tau} = c_n \sim n^{d-2}$$

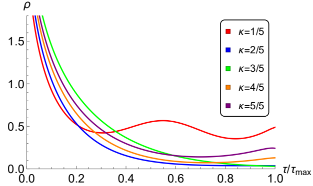
  So the the time to reach infinitely large mode numbers is

$$\Delta\tau = \int^{\infty} \frac{dn}{c_n} \tag{1}$$

  which is log divergent in $d = 3$!

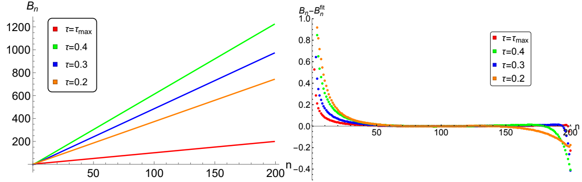
Results: Two-mode data

- For non-equal energy two-mode data ($\kappa \neq 3/5$) we found:



This suggests that there are open sets of initial conditions that collapse and stay regular.

- Finally, the phases become more coherent as ρ decreases:



Conclusions

- Space of initial conditions is more intricate than we initially thought:
  - Both, stability and instability islands survive in the limit $\varepsilon \to 0$ limit
  - Unstable initial conditions can collapse at different time-scales (only in $3+1$?)
- Some open questions and room for improvement:
  - Applications in AdS/CMT? [Abajo-Arrastia, da Silva, Lopez, Mas, Serantes]
  - Break spherical symmetry [Choptuik, Dias, Santos, Way]
  - Include charge [Arias, Mas, Serantes]
  - Understand the next non-linear time-scale / deviations from phase coherence
  - Understand the turbulent instability from first principles