Holographic entanglement entropy in Lovelock theories of gravity with matter (Part II)

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Plan for the talk

1. Entanglement entropy.

2. The Ryu-Takayangi formula.

3.The Lewkowycz-Maldacena method.

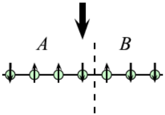
4. Application of the L-M method to Lovelock theories with matter.

5. Conclusions.

Entanglement entropy

If we divide a quantum system defined by a Hilbert space $\mathcal{H}$ in two regions A and B, this space factorizes as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$.

For example, a spin chain:



Total density matrix

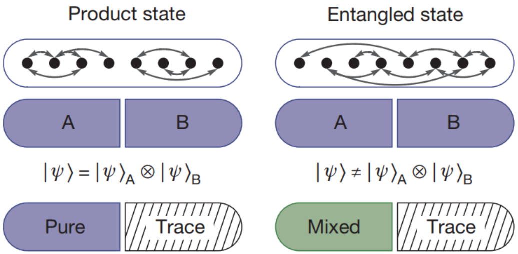
$$\hat{\rho} = \sum_m p_m |\psi_m\rangle\langle\psi_m| \tag{1}$$

Reduced density matrix:

$$\rho_A = Tr_B\ \rho \equiv \sum_j \langle \psi_j^B | \hat{\rho} | \psi_j^B \rangle, \qquad |\psi_j^B\rangle \in \mathcal{H}_B \tag{2}$$

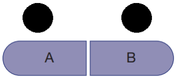
Entanglement entropy:

$$\boxed{S_{EE} = -Tr(\rho_A \log \rho_A)} \tag{3}$$



Entanglement entropy

In the simplest case we have two particles with spin 1/2:



Non-entangled state:

$$|\psi\rangle = \frac{1}{2}(|+\rangle_A + |-\rangle_A) \otimes (|+\rangle_B + |-\rangle_B)$$

$$\rho_A = \frac{1}{2}(|+\rangle_A + |-\rangle_A)((\langle +|_A + \langle -|_A) \implies S_{EE} = 0 \qquad (4)$$

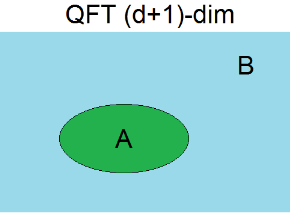
Entangled state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle_A \otimes |-\rangle_B) - (|-\rangle_A \otimes |+\rangle_B)$$

$$\rho_A = \frac{1}{2}(|+\rangle_A\langle +|_A + |-\rangle_A\langle -|_A) \implies S_{EE} = \log 2 \qquad (5)$$

Entanglement entropy

Similarly, we can define the entanglement entropy for a quantum field theory (QFT), which is a relativistic quantum theory with an infinite number of degrees of freedom.



In Feynman's formalism, the sum over states is replaced by a path integral. The partition function is derived from a theory where Euclidean time is periodic:

$$Z[\beta] = Tr[e^{-iH\beta}] \to \int_{\phi(0)=\phi(\beta)} \mathcal{D}\phi e^{-S_E[\phi(x)]} \tag{6}$$

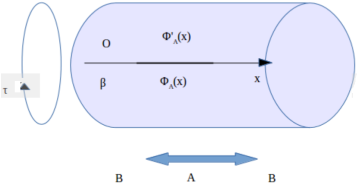
Replica trick

In this formalism, the entanglement entropy $-Tr(\rho_A \log \rho_A)$ becomes

$$S_{EE} = \int_{\phi(0)=\phi(\beta)} \mathcal{D}\phi \langle \phi | \rho_A \log \ \rho_A | \phi \rangle. \quad (7)$$

In general, we don't know how to calculate this integral. Instead, we can define

$$Tr[\rho_A] = Z^{-1} \int_{\tau=-\infty}^{\tau=\infty} \mathcal{D}\phi e^{-S_E[\phi(x)]}$$



(nth)Renyi entropy:

$$S_n = \frac{Tr[\rho_A]^n - 1}{1-n}, \quad (8)$$

[Calabrese & Cardy]

which satisfies

$$S_{EE} = \lim_{n \to 1} S_n = -\frac{\partial}{\partial n} \log Tr[\rho_A^n]|_{n=1} \quad (9)$$

Replica trick

We calculate the product of n reduced density matrices as

$$[\rho_A^n] = [\rho_A]_{\phi_{1+}\phi_{1-}}[\rho_A]_{\phi_{2+}\phi_{2-}}.., [\rho_A]_{\phi_{n+}\phi_{n-}} \tag{10}$$

and join the replicas using the boundary conditions

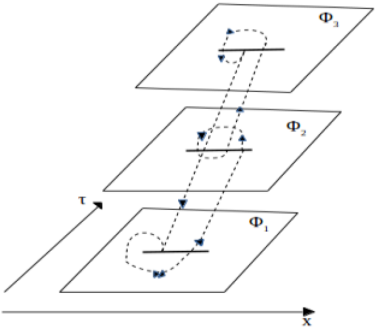
$$\begin{aligned} \phi_j(0^+, x) = \phi_{j+1}(0^-, x) \quad x \in A \\ \phi_j(0^+, x) = \phi_j(0^-, x) \quad x \in B \end{aligned} \tag{11}$$

Then:

$$[\rho_A]_{\phi_+\phi_-} = \frac{1}{Z}\int_{t=-\infty}^{t=\infty} D\phi \quad e^{-S[\phi]} \prod_x \delta(\phi_+(x, \tau = 0) - \phi_+(x)) \times \prod_x \delta(\phi_-(x, \tau = \beta) - \phi_-(x)). \tag{12}$$

Replica trick

We integrate over the replicas, removing the region A by means of the Dirac delta .



$$Tr_A \; \rho_A^n = \frac{1}{(Z)^n} \int_{(t,x)\in\mathcal{R}_n} D\phi \; e^{S_n[\phi]}$$

$$\equiv \frac{Z_n}{(Z)^n}$$

$$S_{EE} = -\lim_{n\to 1} \frac{1}{n-1} (\ln Z_n - n \ln(Z_1)^n)$$

[Calabrese & Cardy]

- Let A be an interval of length l on a (1+1-dim) CFT. Using the properties of the CFT we can get the next result:

$$S_{EE}^{A} = \frac{c}{3}\log(\frac{l}{a}), \tag{13}$$

where c is the central charge, and a is a parameter to regularize the result.

- If this same theory is defined on a manifold with topology S^1, the result is very similar:

$$S_{EE}^{A} = \frac{c}{3}\log\left(\frac{2l}{a}\sin(\theta_0)\right) \tag{14}$$

- A (3+1)-dim CFT, where A is a strip defined by $x_1 \in [-l/2, l/2]$.

$$S_{EE}^{A} = \frac{c}{24\pi}\left[\frac{L^2}{a^2} - \frac{L^2}{l^2}\right] \tag{15}$$

Entanglement entropy

Generic properties

- The EE is UV divergent. This is a consequence of the fact that in a QFT there is an infinite number of degrees of freedom.
- Area law: The dominant divergence is proportional to the *area* of the region in which we calculate the EE;

$$S_A = c_o \frac{Area\{A\}}{\epsilon^{d-1}} + \{Subdominant\ divergences\} + S_{finite} \tag{16}$$

- Satisfies certain inequalities, as the subadditivity conditions:

$$S_{A+B+C} + S_B \geq S_{A+B} + S_{B+C} \tag{17}$$

$$S_A + S_C \geq S_{A+B} + S_{B+C} \tag{18}$$

- Even in the simplest cases, the replica trick becomes a very difficult task to accomplish. Is there an alternative way?

Holographic correspondence

$$\{Type\ IIB\ strings\ on\ AdS_5 \times \mathbb{S}^5\} = \{\mathcal{N} = 4\ SYM\ on\ 4{-}dim\ Minkowski\}$$

[Maldacena]

Classic gravity theory	Quantum field theory
$D+1$ dimensions	D dimensions
Weakly coupled theory	Strongly coupled theory
Asymptotically Anti-de Sitter	Conformal theory (CFT)

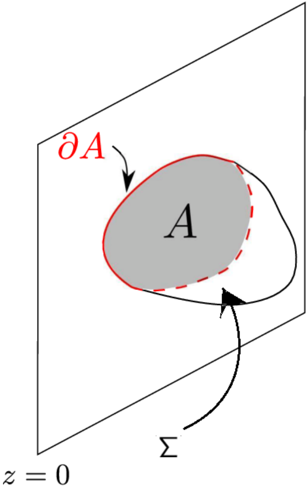
Bulk/boundary correspondence:

$$\boxed{e^{-S_{grav}[\Phi]} = \mathcal{Z}_{CFT}[\phi(x)]} \tag{19}$$

The saclar field Φ defined on the gravity theory has a dual ϕ living in the CFT.

The Ryu-Takayanagi formula

The AdS/CFT correspondence provides us with a prescription to calculate S_{EE} from quantities defined on the gravity side:



$$\boxed{S_{EE}^A = \frac{Area(\Sigma)}{4G}} \qquad (20)$$

[Ryu & Takayanagi]

The surface Σ is obtained by minimizing the area functional:

$$Area = \int d^{d-1}x\sqrt{h} \qquad (21)$$

h is the determinant of the *induced metric*:

$$h_{\alpha\beta} = g_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu \qquad (22)$$

The Ryu-Takayanagi formula

Example: For a metric of the form:

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Sigma_{d-2}^2 \tag{23}$$

we have:

$$S_{EE}^A = \frac{\Omega_{d-2}}{4G}\int_0^{\theta_0} d\theta \sin^{d-2}(\theta) r^{d-2}(\theta)\sqrt{r^2(\theta) + \frac{\partial_\theta r(\theta)^2}{f(r(\theta))}} \tag{24}$$

In a 3D bulk we obtain the result:

$$S_{EE}^A = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh(\pi l/\beta)\right) \tag{25}$$

- The Ryu-Takayanagi formula is valid only if the bulk is Einstein gravity. If we wish to consider theories with higher order terms, this prescription needs to be modified.

Lovelock gravity + conformal matter (brief review)

In $D = 5$, and with a particular choice for s and the couplings, we can rewrite this action as:

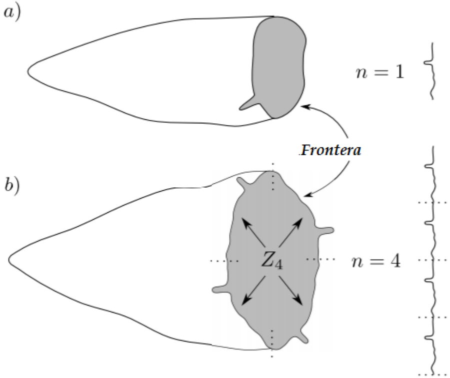
$$
\begin{aligned}
I[g,\phi] = \int_{\mathcal{M}} d^5x \;\; \sqrt{-g}\Big(\frac{\Lambda}{8\pi G} + \frac{R}{16\pi G} + b_0\phi^{25} + b_1\phi^{13}S \\
+ b_2\phi\big(S_{cdab}S^{cdab} - 4S_{ab}S^{ab} + S^2\big)\Big),
\end{aligned}
\tag{26}
$$

Where:

$$
\begin{aligned}
S_{cd}^{\;\;ab} = \phi^2 R_{cd}^{\;\;ab} + \frac{4}{s}\phi \;\; \delta^{[a}_{[c}\nabla_{d]}\nabla^{b]}\phi \;\; + \frac{4(1-s)}{s^2}\delta^{[a}_{[c}\nabla_{d]}\phi\nabla^{b]}\phi \\
- \frac{2}{s^2}\delta^{[a}_{[c}\delta^{b]}_{d]}\nabla_\mu\phi\nabla^\mu\phi.
\end{aligned}
\tag{27}
$$

Lewkowycz-Maldacena (L-M) method

We would like to calculate the EE using the solution presented by Mariano. However, in this theory of gravity, the Ryu-Takayanagi formula is no longer valid, because of the presence of higher-order corrections and the non-minimal couplings.



How do we solve this?
We extend the replica trick to the bulk.
First, we consider a geometry which is a solution to a gravity theory, with a boundary parametrized by the coordinate τ.

L-M method

We construct B_n such that it has the replica manifold, $\mathcal{M}_n$ as its boundary. Using the bulk-boundary correspondence:

$$I[B_n] = -\log(Z[\mathcal{M}_n]) \equiv -\log\ Z_n \tag{28}$$

Now we use the $\mathbb{Z}_n$ symmetry on the replica space to obtain an orbifold which has a periodicity 2π:

$$\hat{B}_n = \frac{B_n}{\mathbb{Z}_n} \tag{29}$$

$$I[\hat{B}_n] = nI[B_n] \tag{30}$$

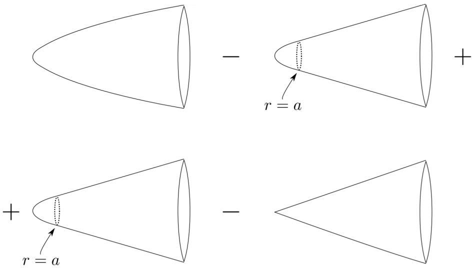
Using the replica trick we find:

$$S_{EE} = \lim_{n\to 1} \frac{n}{n-1}\left(I[\hat{B}_n] - I[\hat{B}_1]\right) = \partial_n I[\hat{B}_n]\Big|_{n=1} \tag{31}$$

L-M method

Regularized squashed cones

As the original period was $2\pi n$, the geometry $\hat{B}_n$ acquire a conic singularity with deficit angle $2\pi/n$. We must regularize this geometry:



$$S_{EE} = -n\partial_n[(\ln Z_n - \ln Z_n^{off}) - (\ln Z_n^{off} - \ln Z_1^n)]_{n=1} \tag{32}$$

[Lewkowycz & Maldacena]

The 4-dim Einstein-Hilbert action is:

$$I_{EH} = \frac{1}{16\pi G_N} \int d^4x \sqrt{g} R \tag{33}$$

We must integrate $\int d^2x \sqrt{g} R$ along the normal coordinates to the codim-2 surface:

$$\int_{Regular \quad Cone} d^2x \sqrt{g} R \sim 4\pi(1-n) \tag{34}$$

From this result we obtain the Ryu-Takayanagi formula, as expected.

$$S_{EE} = \frac{1}{16\pi G_N}(Area)\left(-n\partial_n \int_{Regular \quad Cone} d^2x \sqrt{g} R\right) = \frac{Area}{4G_N} \tag{35}$$

It is convenient to define the small parameter:

$$\epsilon = 1 - \frac{1}{n} \tag{36}$$

In terms of this parameter, the main task is to calculate

$$S_{EE} = \partial_\epsilon I[B_n]|_{\epsilon=0} \tag{37}$$

It is useful to write a Taylor expansion for the metric in terms of the complex coordinates $(z, \bar{z})$, which are normal to the surface:

$$\begin{aligned} ds^2 = {} & e^{2A}[dzd\bar{z} + e^{2A}T(\bar{z}dz - zd\bar{z})^2] + (g_{ij} + 2K_{aij}x^a \\ & + Q_{abij}x^a x^b)dy^i dy^j + 2ie^{2A}(U_i + V_{ai}x^a)(\bar{z}dz - zd\bar{z})dy^i + \dots, \end{aligned} \tag{38}$$

and analogously for the scalar field:

$$\phi = \phi^{(o)} + \phi_z z + \phi_{\bar{z}}\bar{z} + \phi_{zz}z^2 + \phi_{\bar{z}\bar{z}}\bar{z}^2 + \phi_{z\bar{z}}(z\bar{z})^{1-\epsilon} + \dots \tag{39}$$

L-M method

Summary:

- Substitute (38) y (39) into the gravity action and isolate the terms to first order in ϵ.
- Integrate along the normal coordinates to the codim-2 surface, using a as a cut parameter and subtract the singular geometry.
- Take the limit $a \to 0$.
- Additionally, we must find the equations of the codim-2 surface. This is achieved by replacing the previous expansion in the eom's, and imposing the condition that the divergences in the gravity part canceled out.

Can we apply this method when there are non-minimal couplings?

- Using this method, it has been achieved to calculate holographic entanglement entropy in several examples of theories that include higher order corrections.
- Currently, in the literature there are just a few examples where the functional S_{EE} is calculated in theories with non-minimal couplings. [Caceres, Mohan & Nguyen]
- The previously studied gravity theory includes higher order corrections, as well as non-minimal couplings, making it a novel scenario to apply the L-M method.

We separate the action as:

$$I[g_{\mu\nu},\phi] = I_{EH} + I_R + I_H + I_{H'} + I_{GB} + I_M, \qquad (40)$$

where

$$I_R = \beta_1 \int_{\mathcal{M}} d^5x\sqrt{-g}\ \phi^{15} R, \qquad (41)$$

$$I_H = -480\beta_2 \int_{\mathcal{M}} d^5x\sqrt{-g}\ (R_{ab} + \frac{1}{8}g_{ab}R)\ \phi^3\nabla^a\phi\nabla^b\phi, \qquad (42)$$

$$I_{H'} = 80\beta_2 \int_{\mathcal{M}} d^5x\sqrt{-g}\ (R_{ab} - \frac{1}{2}g_{ab}R)\ \phi^4\nabla^a\nabla^b\phi, \qquad (43)$$

$$I_{GB} = \beta_2 \int_{\mathcal{M}} d^5x\sqrt{-g}\ \phi^5\left(R_{cdab}R^{cdab} - 4R_{ab}R^{ab} + R^2\right), \qquad (44)$$

Application of the L-M method to our scenario.

The functional S_{EE} will receive contributions from each of the terms in the action, and takes the form:

$$S_{EE} = S_{EH} + S_R + S_H + S_{H'} + S_{GB} + S_M \tag{45}$$

As explained before, we must look for the linear terms in epsilon in each of these terms For example, we know that the Einstein-Hilbert term, contributes to the functional as:

$$S_{EH} = \frac{A}{4G_N} = \frac{1}{4G_N} \int d^3y\sqrt{h}. \tag{46}$$

Easily, we can obtain the contribution from I_R:

$$S_R = \frac{\beta_1}{4G_N} \int d^3y(\sqrt{h}\ \phi^{15}). \tag{47}$$

Application of the L-M method to our scenario

For the rest of the contributions, we need to know the form of the expansion for an arbitrary power of the scalar field. The result is :

$$\phi^n = (\phi^{(o)})^n + \sum_{l=0}^{k} \sum_{k=0}^{n-1} C_{lk} \rho^{2((l+n-k)-\epsilon(n-k))} + \ldots \tag{48}$$

For instance, from the Dong-Camps formula, we know the form of the functional for GB gravity without the ϕ^5 factor:

$$S_{GB} = 4\pi\beta_2 \left(\int_\Sigma d^3y \sqrt{h} \mathcal{R} \right), \tag{49}$$

where $\mathcal{R}$ is the intrinsic scalar curvature on the codimension 2 surface.

Application of the L-M method to our scenario

Using (48), we can see that only the first term in the expansion of ϕ (a zero order), gives a linear term in ϵ after substituting it in the action I_{GB}. The result is given by:

$$S_{GB} = 4\pi\beta_2 \left(\int_{\Sigma} d^3y \sqrt{h} \mathcal{R} \phi^5 \right) , \tag{50}$$

For this term, we additionally verify that there are no terms to quadratic order in ϵ that may produce an anomalous term. For the

variational principle to be well defined, we need to consider a boundary term as well.

Results

Following the same logic, we can deduce the contributions corresponding to I_H and $I_{H'}$. In the end, the complete functional is:

$$S_{EE} = \int_\Sigma d^3y\sqrt{h}\Big(\frac{1}{4G} \; + 4\pi\big(\beta_1\phi^{15} + \beta_2\mathcal{R}\phi^5 - 60\beta_2\phi^3 h^{ij}\;\partial_i\phi\partial_j\phi$$
$$- 40\beta_2\phi^4 h^{ij}\partial_i\partial_j\phi\big)\Big) + 2\beta_2\int_{\partial\Sigma}\phi^5 K, \qquad (51)$$

Where the last term is an appropriate boundary term to evaluate the functional on the black hole solution of this theory. Using explicitly our solution for $g_{\mu\nu}$ and ϕ, we can calculate, for example, the holographic entanglement entropy of a spherical region.

Conclusions and future work

- The contributions of the action to the EE appear as variants of well-studied gravity theories and where the corresponding HEE functional has been obtained, such as Horndeski's gravity and Gauss-Bonnet gravity
- The holographic entanglement entropy functional receives contributions from all these term, and the power of the scalar field that appeared in the action is inherited to the functional.
- Although it is something that has recently been demonstrated as true in general [Dong & Lewkowycz: arXiv:1705.08453], it is necessary to demonstrate in detail that the extremality condition on the functional that we have obtained is in fact the condition that is obtained by requiring that there are no divergences in the equations of motion.